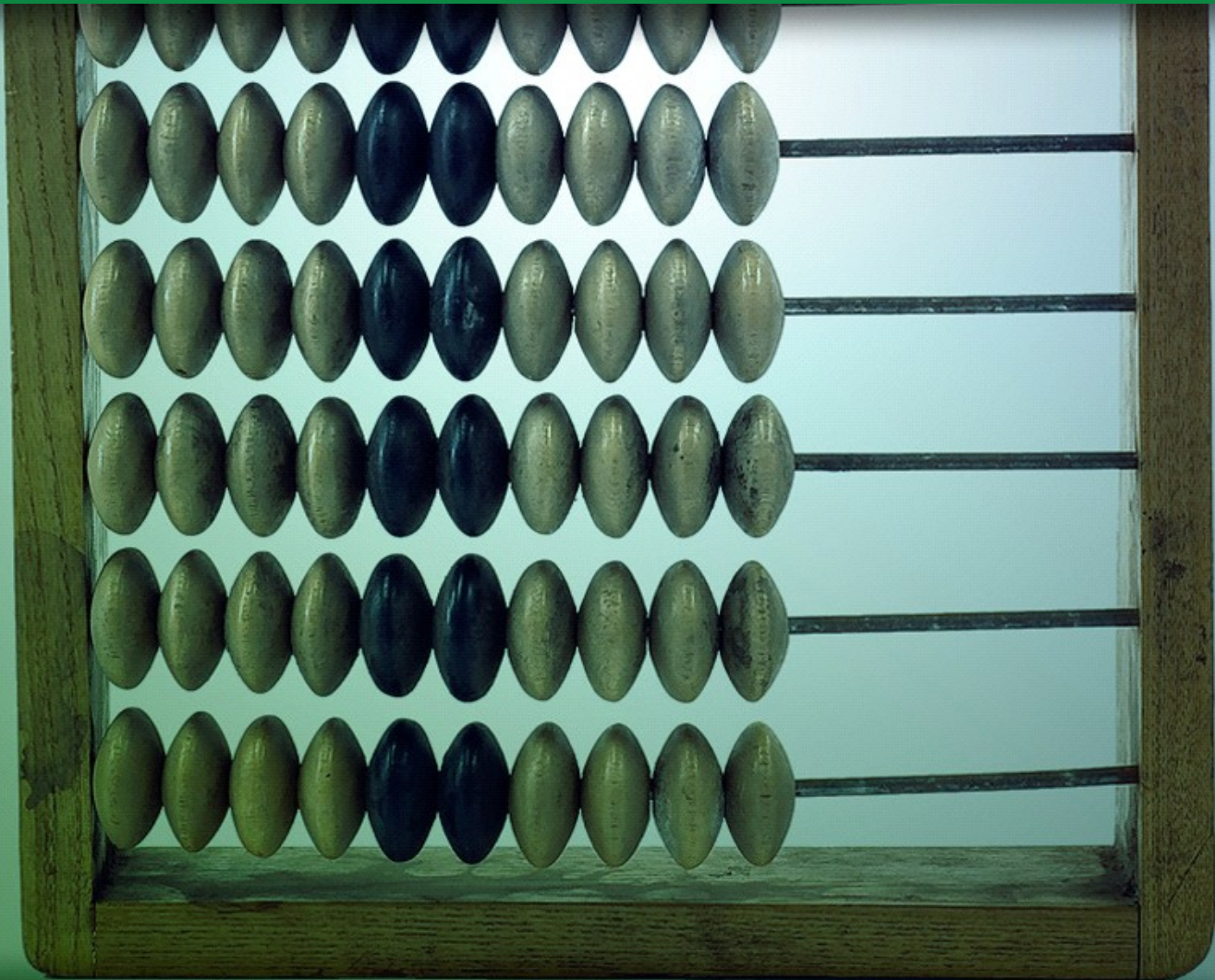


LEIF MEJLBRO

**REAL FUNCTIONS OF SEVERAL VARIABLES -
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Leif Mejlbro

Real Functions of Several Variables

Examples of Applications of Gauß's and Stokes's Theorems and Related Topics

Calculus 2c-9

Real Functions of Several Variables – Examples of Applications of Gauß’s and Stokes’s
Theorems and Related Topics – Calculus 2c-9
© 2007 Leif Mejlbro & Ventus Publishing ApS
ISBN 978-87-7681-261-4

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Preface

In this volume I present some examples of *applications of Gauß's and Stokes's theorems and related topics*, cf. also *Calculus 2b, Functions of Several Variables*. Since my aim also has been to demonstrate some solution strategy I have as far as possible structured the examples according to the following form

A *Awareness*, i.e. a short description of what is the problem.

D *Decision*, i.e. a reflection over what should be done with the problem.

I *Implementation*, i.e. where all the calculations are made.

C *Control*, i.e. a test of the result.

This is an ideal form of a general procedure of solution. It can be used in any situation and it is not linked to Mathematics alone. I learned it many years ago in the Theory of Telecommunication in a situation which did not contain Mathematics at all. The student is recommended to use it also in other disciplines.

One is used to from high school immediately to proceed to **I. Implementation**. However, examples and problems at university level are often so complicated that it in general will be a good investment also to spend some time on the first two points above in order to be absolutely certain of what to do in a particular case. Note that the first three points, **ADI**, can always be performed.

This is unfortunately not the case with **C Control**, because it from now on may be difficult, if possible, to check one's solution. It is only an extra securing whenever it is possible, but we cannot include it always in our solution form above.

I shall on purpose not use the logical signs. These should in general be avoided in Calculus as a shorthand, because they are often (too often, I would say) misused. Instead of \wedge I shall either write "and", or a comma, and instead of \vee I shall write "or". The arrows \Rightarrow and \Leftrightarrow are in particular misunderstood by the students, so they should be totally avoided. Instead, write in a plain language what you mean or want to do.

It is my hope that these examples, of which many are treated in more ways to show that the solutions procedures are not unique, may be of some inspiration for the students who have just started their studies at the universities.

Finally, even if I have tried to write as careful as possible, I doubt that all errors have been removed. I hope that the reader will forgive me the unavoidable errors.

Leif Mejlbro
15th October 2007

1 Gradient fields and integrals

Example 1.1 Check in each of the following cases if the given vector field

$$\mathbf{V} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

is a gradient field. Whenever this is the case one shall also find all the integrals of the gradient field.

1) $\mathbf{V}(x, y, z) = (2xz \exp(x^2 + y^2), 2yz \exp(x^2 + y^2), \exp(x^2 + y^2)).$

2) $\mathbf{V}(x, y, z) = (x + z, -y - z, x - y).$

3) $\mathbf{V}(x, y, z) = (2xy^3, x^2z^3, 3x^2yz^2).$

4) $\mathbf{V}(x, y, z) = (z^3 + y^2 \cos x, 2y \sin x - 4, 3xz^2 + 2).$

5) $\mathbf{V}(x, y, z) = (3y^4z^2, 4x^3z^2, -3x^2y^2).$

6) $\mathbf{V}(x, y, z) = (4xy - 3x^2z^2 + 1, 2x^2 + 2, -2x^3z - 3z^2).$

7) $\mathbf{V}(x, y, z) = (2x^2 + 8xy^2, 3x^3y - 3xy, -4y^2z^2 - 2x^3z).$

8) $\mathbf{V}(x, y, z) = (y \cosh(xy), z + x \cosh(xy), y).$

9) $\mathbf{V}(x, y, z) = (e^y, e^x, x + y + z).$

10) $\mathbf{V}(x, y, z) = \frac{1}{1 + x^2y^2 + 2xyz^2 + z^4} (y, x, 2z).$

A Gradient fields.

D Check directly by some manipulation of the rules of calculation if $\mathbf{V} \cdot d\mathbf{x}$ is a total differential.

ALTERNATIVELY one integrates along a broken line \mathcal{K} from $(0, 0, 0)$, and then check by taking the gradient of the result and compare with \mathbf{V} .

I 1) We get by some manipulation

$$\begin{aligned} \mathbf{V} \cdot d\mathbf{x} &= 2xz \exp(x^2 + y^2)dx + 2yz \exp(x^2 + y^2)dy + \exp(x^2 + y^2)dz \\ &= z \exp(x^2 + y^2) \{2x dx + 2y dy\} + \exp(x^2 + y^2)dz \\ &= z \exp(x^2 + y^2) d(x^2 + y^2) + \exp(x^2 + y^2)dz \\ &= z d(\exp(x^2 + y^2)) + 1 \cdot \exp(x^2 + y^2)dz \\ &= d(z \exp(x^2 + y^2)), \end{aligned}$$

which shows that \mathbf{V} is a gradient field and that all its integrals are given by

$$\Phi(x, y, z) = z \exp(x^2 + y^2) + k, \quad k \in \mathbb{R}.$$

ALTERNATIVELY, let \mathcal{K} be the broken line

$$(0, 0, 0) \longrightarrow (x, 0, 0) \longrightarrow (x, y, 0) \longrightarrow (x, y, z).$$

By a tangential line integral along this curve we get using that $z = 0$ along the first two segments,

$$\Phi(x, y, z) = \int_0^x 0 dt + \int_0^y 0 dt + \int_0^z \exp(x^2 + y^2) dz = z \exp(x^2 + y^2)$$

and the gradient is

$$\nabla\Phi = (2xz \exp(x^2+y^2), 2yz \exp(x^2+y^2), \exp(x^2+y^2)) = \mathbf{V}(x, y, z),$$

showing that \mathbf{V} is a gradient field and that all integrals are given by

$$\Phi(x, y, z) = z \exp(x^2 + y^2) + k, \quad k \in \mathbb{R}.$$

2) By a small manipulation,

$$\begin{aligned} \mathbf{V} \cdot d\mathbf{x} &= (x+z)dx - (y+z)dy + (x-y)dz \\ &= d\left(\frac{1}{2}x^2\right) - d\left(\frac{1}{2}y^2\right) + (z dx + x dz) - (z dy + y dz) \\ &= d\left(\frac{1}{2}x^2 - \frac{1}{2}y^2 + xz - yz\right), \end{aligned}$$

proving that \mathbf{V} is a gradient field with the integrals

$$\begin{aligned} \Phi(x, y, z) &= \frac{1}{2}x^2 - \frac{1}{2}y^2 + z(x-y) + k \\ &= \frac{1}{2}(x-y)(x+y+2z) + k, \quad \text{for } k \in \mathbb{R}. \end{aligned}$$

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ALTERNATIVELY, we get by a line integration along the usual broken line from $(0, 0, 0)$,

$$\begin{aligned}\Phi(x, y, z) &= \int_{\mathcal{K}} \mathbf{V} \cdot d\mathbf{x} = \int_0^x t \, dt - \int_0^y t \, dt + \int_0^z (x - y) \, dt \\ &= \frac{1}{2}x^2 - \frac{1}{2}y^2 + z(x - y),\end{aligned}$$

with the gradient

$$\nabla\Phi = (x + z, -y - z, x - y) = \mathbf{V}(x, y, z).$$

This proves that \mathbf{V} is a gradient field, and the integrals are

$$\Phi(x, y, z) = \frac{1}{2}x^2 - \frac{1}{2}y^2 + z(x - y) + k, \quad k \in \mathbb{R}.$$

3) Since e.g.

$$\frac{\partial V_1}{\partial y} = 6xy^2 \neq \frac{\partial V_2}{\partial x} = 2xz^3,$$

the *necessary* condition for a gradient field is not fulfilled, so \mathbf{V} is not a gradient field.

ALTERNATIVELY, let us see what happens if we instead use the method of integrating along the usual broken line from $(0, 0, 0)$:

$$\Phi(x, y, z) = \int_0^x 0 \, dt + \int_0^y 0 \, dt + 3x^2y \int_0^z t^2 \, dt = x^2yz^3.$$

The gradient of this result is

$$\nabla\Phi = (2xyz^3, x^2z^3, 3x^2yz^2) \neq \mathbf{V}(x, y, z),$$

because the two fields do not agree in their first coordinate. Hence, the vector field \mathbf{V} is not a gradient field.

4) By a small manipulation,

$$\begin{aligned}\mathbf{V} \cdot d\mathbf{x} &= (z^3 + y^2 \cos x)dx + (2y \sin x - 4)dy + (3xz^2 + 2)dz \\ &= (z^3 dx + x \cdot 3z^2 dz) + (y^2 \cos x dx + \sin x \cdot 2y dy) - 4 dy + 2 dz \\ &= d(xz^3 + y^2 \sin x - 4y + 2z),\end{aligned}$$

proving that \mathbf{V} is a gradient field with the integrals

$$\Phi(x, y, z) = xz^3 + y^2 \sin x - 4y + 2z + k, \quad k \in \mathbb{R}.$$

ALTERNATIVELY, we get by integrating along the usual broken line from $(0, 0, 0)$,

$$\begin{aligned}\Phi(x, y, z) &= \int_0^x 0 \, dt + \int_0^y (2t \sin x - 4) \, dt + \int_0^z (3xt^2 + 2) \, dt \\ &= y^2 \sin x - 4y + xz^3 + 2z.\end{aligned}$$

The gradient of this result is

$$\nabla\Phi = (y^2 \cos x + z^3, 2y \sin x - 4, 3xz^2 + 2) = \mathbf{V}(x, y, z),$$

so the vector field is a gradient field, and its integrals are

$$\Phi(x, y, z) = xz^3 + y^2 \sin x - 4y + 2z + k, \quad k \in \mathbb{R}.$$

5) Since e.g.

$$\frac{\partial V_1}{\partial y} = 12y^3 z^2 \quad \text{og} \quad \frac{\partial V_2}{\partial x} = 12x^2 z^2,$$

we have

$$\frac{\partial V_1}{\partial y} \neq \frac{\partial V_2}{\partial x},$$

and the *necessary* condition for an integral is not satisfied. This proves that the vector field is not a gradient field.

ALTERNATIVELY we get by integrating along the usual broken line from $(0, 0, 0)$,

$$\Phi(x, y, z) = \int_0^x 0 dt + \int_0^y 0 dt - 3x^2 y^2 \int_0^z dt = -3x^2 y^2 z.$$

The gradient of this result is

$$\nabla \Phi = (-6xy^2 z, -6x^2 y z, -3x^2 y^2) \neq \mathbf{V}(x, y, z).$$

It follows that the vector field is not a gradient field.

6) By a small manipulation,

$$\begin{aligned} \mathbf{V} \cdot d\mathbf{x} &= (4xy - 3x^2 z^2 + 1)dx + (2x^2 + 2)dy + (-2x^3 z - 3z^2)dz \\ &= (4xy dx + 2x^2 dy) - (3x^2 z^2 dx + 2x^3 z dz) + d(x + 2y - z^3) \\ &= 2\{y d(x^2) + x^2 dy\} - \{z^2 d(x^3) + x^3 d(z^2)\} + d(x + 2y - z^3) \\ &= d(x + 2y - z^3 + 2x^2 y - x^3 z^2), \end{aligned}$$

so the vector field is a gradient field with the integrals

$$\Phi(x, y, z) = x + 2y - z^3 + 2x^2 y - x^3 z^2 + k, \quad k \in \mathbb{R}.$$

ALTERNATIVELY we integrate along the usual broken line \mathcal{K} from $(0, 0, 0)$,

$$\begin{aligned} \Phi(x, y, z) &= \int_0^x dt + \int_0^y (2x^2 + 2)dt - \int_0^z (2x^3 t + 3t^2)dt \\ &= x + 2x^2 y + 2y - x^3 z^2 - z^3. \end{aligned}$$

The gradient of this result is

$$\nabla \Phi = (1 + 4xy - 3x^2 z^2, 2x^2 + 2, -2x^3 z - 3z^2) = \mathbf{V}(x, y, z),$$

so the vector field \mathbf{V} is a gradient field with the integrals

$$\Phi(x, y, z) = x + 2y - z^3 + 2x^2 y - x^3 z^2 + k, \quad k \in \mathbb{R}.$$

7) It follows from

$$\frac{\partial V_1}{\partial y} = 16xy \neq \frac{\partial V_2}{\partial x} = 9x^2 y - 3y,$$

that the *necessary* condition is not fulfilled, so the vector field is not a gradient field.

ALTERNATIVELY we integrate along the broken line \mathcal{K} ,

$$\begin{aligned}\Phi(x, y, z) &= \int_0^x (2t^2 + 0)dt + \int_0^y (3x^3t - 3xt)dt + \int_0^z (-4y^2t^2 - 2x^3t)dt \\ &= \frac{2}{3}x^3 + \frac{3}{2}x^3y^2 - \frac{3}{2}xy^2 - \frac{4}{3}y^2z^3 - x^3z^2.\end{aligned}$$

The gradient of this expression is

$$\begin{aligned}\nabla\Phi &= \left(2x^2 + \frac{9}{2}x^2y^2 - \frac{3}{2}y^2 - 3x^2z^2, 3x^3y - 3xy - \frac{8}{3}yz^3, -4y^2z^2 - 2x^3z\right) \\ &\neq \mathbf{V}(x, y, z).\end{aligned}$$

We see that the *necessary* condition is not satisfied, so \mathbf{V} is not a gradient field.

8) By a small manipulation,

$$\begin{aligned}\mathbf{V} \cdot d\mathbf{x} &= y \cosh(xy)dx + (z + x \cosh(xy))dy + y dz \\ &= \cosh(xy)\{y dx + x dy\} + \{z dy + y dz\} \\ &= \cosh(xy) d(xy) + d(yz) \\ &= d\{\sinh(xy) + yz\},\end{aligned}$$

which shows that \mathbf{V} is a gradient field with the integrals given by

$$\Phi(x, y, z) = \sinh(xy) + yz + k, \quad k \in \mathbb{R}.$$

ALTERNATIVELY we integrate along the usual broken line \mathcal{K} ,

$$\Phi(x, y, z) = \int_0^x 0 dt + \int_0^y x \cosh(xt)dt + \int_0^z y dt = \sinh(xy) + yz.$$

The gradient of this expression is

$$\nabla\Phi = (y \cosh(xy), z + x \cosh(xy), y) = \mathbf{V}(x, y, z).$$

This shows that \mathbf{V} is a gradient field and that all the integrals are given by

$$\Phi(x, y, z) = \sinh(xy) + yz + k, \quad k \in \mathbb{R}.$$

9) It follows from

$$\frac{\partial V_1}{\partial y} = e^y \neq \frac{\partial V_2}{\partial x} = e^x,$$

that the *necessary* condition is not fulfilled, so \mathbf{V} is not a gradient field.

ALTERNATIVELY an integration along \mathcal{K} gives

$$\begin{aligned}\Phi(x, y, z) &= \int_0^x e^0 dt + \int_0^y e^x dt + \int_0^z (x + y + t)dt \\ &= x + y e^x + (x + y)z + \frac{1}{2}z^2.\end{aligned}$$

The gradient of this expression is

$$\nabla\Phi = (1 + y e^x + z, e^x + z, x + y + z) \neq \mathbf{V}(x, y, z).$$

As the *necessary* condition is not fulfilled, \mathbf{V} is not a gradient field.

10) We get by a small manipulation,

$$\begin{aligned}\mathbf{V} \cdot d\mathbf{x} &= \frac{1}{1 + x^2y^2 + 2xyz^2 + z^4} (y dx + x dy + 2z dz) \\ &= \frac{1}{1 + (xy)^2 + 2 \cdot xy \cdot z^2 + (z^2)^2} d(xy + z^2) \\ &= \frac{1}{1 + (xy + z^2)^2} d(xy + z^2) = d\text{Arctan}(xy + z^2).\end{aligned}$$

This shows that \mathbf{V} is a gradient field with the integrals

$$\Phi(x, y, z) = \text{Arctan}(xy + z^2) + k, \quad k \in \mathbb{R}.$$

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ALTERNATIVELY the integration along \mathcal{K} gives

$$\begin{aligned}\Phi(x, y, z) &= \int_0^x 0 \, dt + \int_0^y \frac{x}{1+x^2t^2} \, dt + \int_0^z \frac{2t}{1+x^2y^2+2xyt^2+t^4} \, dt \\ &= \operatorname{Arctan}(xy) + \int_{t=0}^z \frac{1}{1+(xy+t^2)^2} \, d(t^2) \\ &= \operatorname{Arctan}(xy) + [\operatorname{Arctan}(xy+t^2)]_{t=0}^z \\ &= \operatorname{Arctan}(xy) + \operatorname{Arctan}(xy+z^2) - \operatorname{Arctan}(xy) \\ &= \operatorname{Arctan}(xy+z^2).\end{aligned}$$

It follows from

$$\nabla\Phi = \frac{1}{1+(xy+z^2)^2} (y, x, 2z) = \frac{1}{1+x^2y^2+2xyt^2+t^4} (y, x, 2z) = \mathbf{V}(x, y, z),$$

that \mathbf{V} is a gradient field with the integrals given by

$$\Phi(x, y, z) = \operatorname{Arctan}(xy+z^2) + k, \quad k \in \mathbb{R}.$$

Example 1.2 Consider the vector field

$$\mathbf{V}(x, y) = \left(\frac{2x+y}{x^2+xy+y^2} + ye^{xy}, \frac{x+2y}{x^2+xy+y^2} + xe^{xy} \right), \quad (x, y) \neq (0, 0).$$

Prove that \mathbf{V} is a gradient field in the set given by $y > -|x|$ and find all its integrals

Then prove that \mathbf{V} is a gradient field in its full domain.

Finally find the tangential line integral of \mathbf{V} along the broken line from $(2, 0)$ via the points $(2, 2)$ and $(-2, 2)$ to $(-2, 0)$.

A Gradient field.

D Either guess an integral, or integrate along a broken line from $(0, 2)$. Check the result. Finally, use this integral to find the tangential line integral of \mathbf{V} along this broken line.

I For $(x, y) \neq (0, 0)$ it follows by a small manipulation,

$$\begin{aligned}\mathbf{V} \cdot d\mathbf{x} &= \left(\frac{2x+y}{x^2+xy+y^2} + ye^{xy} \right) dx + \left(\frac{x+2y}{x^2+xy+y^2} + xe^{xy} \right) dy \\ &= \frac{1}{x^2+xy+y^2} \{ (2x+y)dx + (x+2y)dy \} + e^{xy} \{ y dx + x dy \} \\ &= \frac{1}{x^2+xy+y^2} d(x^2+xy^2) + e^{xy} d(xy) \\ &= d(e^{xy} + \ln(x^2+xy^2)).\end{aligned}$$

Hence, \mathcal{V} is a gradient field in its full domain, and its integrals are given by

$$\Phi(x, y) = e^{xy} + \ln(x^2+xy+y^2) + k, \quad k \in \mathbb{R}.$$

ALTERNATIVELY we get by integrating along the broken line

$$(0, 2) \longrightarrow (x, 2) \longrightarrow (x, y),$$

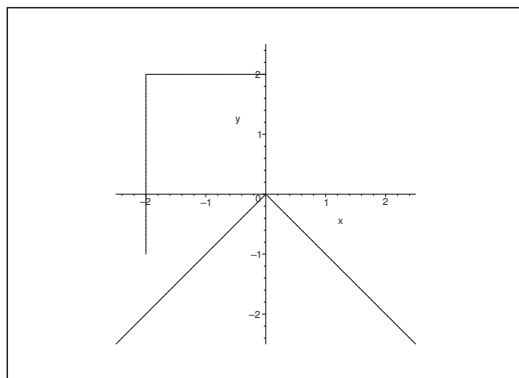


Figure 1: The path of integration in the domain given by $y > -|x|$.

assuming that (x, y) does not lie on the negative Y -axis (actually a larger domain than the given).

$$\begin{aligned} \Phi(x, y) &= \int_0^x \left\{ \frac{2t+2}{t^2+2t+4} + 2e^{2t} \right\} dt + \int_2^y \left\{ \frac{2t+x}{x^2+xt+t^2} + x e^t \right\} dt \\ &= [\ln(t^2+2t+4)]_0^x + [e^{2t}]_0^x + [\ln(x^2+xt+t^2)]_2^y + [e^{xt}]_2^y \\ &= \ln(x^2+2x+4) - \ln 4 + e^{2x} - 1 + \ln(x^2+xy+y^2) - \ln(x^2+2x+4) + e^{xy} - e^{2x} \\ &= e^{xy} + \ln(x^2+xy+y^2) - 1 - 2 \ln 2. \end{aligned}$$

It follows from

$$(1) \quad \nabla \Phi = \left(\frac{2x+y}{x^2+xy+y^2} + y e^{xy}, \frac{x+2y}{x^2+xy+y^2} + x e^{xy} \right) = \mathbf{V}(x, y),$$

that \mathbf{V} is a gradient field. Now, the expression of Φ is defined for $(x, y) \neq (0, 0)$, and (1) holds in this domain, thus we conclude that \mathbf{V} is a gradient field for $(x, y) \neq (0, 0)$.

The integrals are then given by

$$\Phi(x, y) = e^{xy} + \ln(x^2+xy+y^2) + k, \quad k \in \mathbb{R}.$$

Let \mathcal{K} be the broken line

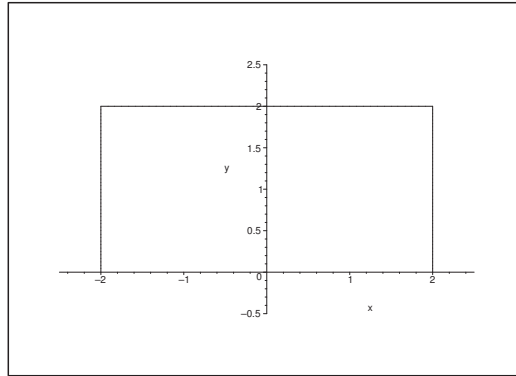
$$(2, 0) \longrightarrow (2, 2) \longrightarrow (-2, 2) \longrightarrow (-2, 0).$$

Then

$$\int_{\mathcal{K}} \mathbf{V} \cdot d\mathbf{x} = \Phi(-2, 0) - \Phi(2, 0) = e^{-2 \cdot 0} + \ln(4+0) - 2^{2 \cdot 0} - \ln(4+0) = 0.$$

ALTERNATIVELY and more difficult a parametric description of \mathcal{K} is given by

$$\mathbf{r}(t) = \begin{cases} (2, 2t), & t \in [0, 1], \\ (6-4t, 2), & t \in [1, 2], \\ (-2, 6-2t), & t \in [2, 3], \end{cases}$$

Figure 2: The path of integration from $(2, 0)$ to $(-2, 0)$.

where

$$\mathbf{r}'(t) = \begin{cases} (0, 2), & t \in]0, 1[, \\ (-4, 0), & t \in]1, 2[, \\ (0, -2), & t \in]2, 3[, \end{cases}$$

so

$$\begin{aligned} \int_{\mathcal{K}} \mathbf{V} \cdot d\mathbf{x} &= \int_0^1 \left\{ \frac{2+4t}{4+4t+4t^2} + 2e^{4t} \right\} 2 dt \\ &\quad + \int_1^2 \left\{ \frac{2(6-4t)+2}{(6-4t)^2+2(6-4t)+4} + 2e^{2(6-4t)} \right\} (-4) dt \\ &\quad + \int_2^3 \left\{ \frac{-2+2(6-2t)}{4-2(6-2t)+(6-2t)^2} - 2e^{-2(6-2t)} \right\} (-2) dt \\ &= [\ln(4+4t+4t^2)]_0^1 + [e^{4t}]_0^1 \\ &\quad + [\ln((6-4t)^2+2(6-4t)+4)]_1^2 + [e^{2(6-4t)}]_1^2 + 2 \\ &\quad + [\ln(4-2(6-2t)+(6-2t)^2)]_2^3 + [e^{-2(6-2t)}]_2^3 \\ &= \ln(12) - \ln 4 + e^4 - 1 \\ &\quad + \ln(4-4+4) - \ln(4+4+4) + e^{-4} - e^4 \\ &\quad + \ln 4 - \ln(4-4+4) + 1 - e^{-4} \\ &= \{\ln 12 - \ln 4 + e^4 - 1\} + \{\ln 4 - \ln 12 + e^{-4} - e^4\} + \{\ln 4 - \ln 4 + 1 - e^{-4}\} \\ &= 0. \end{aligned}$$

REMARK. The computations would have been easier if we instead had considered the parametric description

$$\mathbf{r}(t) = \begin{cases} (2, t), & t \in [0, 2], \\ (-t, 2), & t \in [-2, 2], \\ (-2, 2-t), & t \in [0, 2]. \end{cases} \quad \diamond$$

Example 1.3 Prove that the vector field


$$\mathbf{V}(x, y, z) = \left(\frac{2xz^2 - y}{x^2 + y^2}, \frac{2yz^2 + x}{x^2 + y^2}, 2z \ln(x^2 + y^2) \right), \quad (x, y) \neq (0, 0),$$

is a gradient field in the set $A = \{(x, y, z) \mid x > 0\}$, and find the integral $F : A \rightarrow \mathbb{R}$, for which $F(1, 0, 0) = 0$. Then compute the tangential line integral of \mathbf{V} along the curve \mathcal{K} given by

$$(x, y, z) = \left(1, t^2, \frac{t}{\sqrt{2}} \right), \quad t \in [0, 1].$$

Finally, show that \mathbf{V} is not a gradient field in its full domain. (Compute the circulation along some circle in a plane perpendicular to the Z -axis).

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A Gradient field.

D The integral is found by one of the following three methods:

First method. Rules of calculations for differentials.

Second method. Indefinite integration.

Third method. Integration along a curve (e.g. a broken line) followed by the mandatory check.

The tangential line integral is found by:

First method. Use the definition of the integral.

Second method. Insert the parametric description and just compute-

Finally, compute the circulation along the unit circle. If the value is different from 0, the vector field \mathbf{V} cannot be a gradient field in the whole of its domain.

REMARK. The latter result illustrates the importance of the domain being *simply connected*. \diamond

I *The integral.*

First method. Rules of calculations for differentials. If $x > 0$, then

$$\begin{aligned} \mathbf{V} \cdot d\mathbf{x} &= \frac{2xz^2 - y}{x^2 + y^2} dx + \frac{2yz^2 + x}{x^2 + y^2} dy + 2z \ln(x^2 + y^2) dz \\ &= \left\{ \frac{z^2}{x^2 + y^2} (2x dx + 2y dy) + \ln(x^2 + y^2) d(z^2) \right\} + \frac{1}{x^2 + y^2} (-y dx + x dy) \\ &= \left\{ z^2 \frac{d(x^2 + y^2)}{x^2 + y^2} + \ln(x^2 + y^2) d(z^2) \right\} + \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{-y dx + x dy}{x^2} \\ &= \left\{ z^2 d \ln(x^2 + y^2) + \ln(x^2 + y^2) d(z^2) \right\} + \frac{1}{1 + \left(\frac{y}{x}\right)^2} d\left(\frac{y}{x}\right) \\ &= d \left\{ z^2 \ln(x^2 + y^2) + \operatorname{Arctan} \left(\frac{y}{x} \right) \right\}. \end{aligned}$$

This shows that

$$\mathbf{V} \cdot d\mathbf{x} = dF = \nabla F \cdot d\mathbf{x},$$

hence \mathbf{V} is a gradient field with the integrals given by

$$F_k(x, y, z) = z^2 \ln(x^2 + y^2) + \operatorname{Arctan} \left(\frac{y}{x} \right) + k, \quad k \in \mathbb{R}, \quad (x, y, z) \in A.$$

It follows from the condition $F(1, 0, 0) = 0 + 0 + k = 0$ that $k = 0$, thus

$$F(x, y, z) = F_0(x, y, z) = z^2 \ln(x^2 + y^2) + \operatorname{Arctan} \left(\frac{y}{x} \right), \quad x > 0.$$

Second method. Successive indefinite integration. Let again $x > 0$ and put

$$\begin{aligned} \omega &= \mathbf{V} \cdot d\mathbf{x} \\ &= \frac{2xz^2 - y}{x^2 + y^2} dx + \frac{2yz^2 + x}{x^2 + y^2} dy + 2z \ln(x^2 + y^2) dz. \end{aligned}$$

An inspection shows that the latter term will give less trouble by indefinite integration, so we put

$$F_1(x, y, z) = \int 2z \ln(x^2 + y^2) dz = z^2 \ln(x^2 + y^2).$$

Then

$$dF_1 = \frac{2xz^2}{x^2 + y^2} dx + \frac{2yz^2}{x^2 + y^2} dy + 2z \ln(x^2 + y^2) dz,$$

and

$$(2) \quad \omega - dF_1 = \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy.$$

We shall be more careful in the next step. If we choose the first term, we shall usually get an Arctan-function with y in the denominator. However, y may be 0, so we get some difficulties in finding a continuous integral. There is here a trick to circumvent this hurdle which is not too well-known, so we shall show it here aside.

ASIDE. Since $x > 0$ we may put

$$\begin{aligned} \Phi_2(x, y) &= \int \frac{-y}{x^2 + y^2} dx = \int \frac{1}{1 + \left(\frac{y}{x}\right)^2} \left(-\frac{y}{x^2}\right) dx \\ &= \int \frac{1}{1 + \left(\frac{y}{x}\right)^2} d\left(\frac{y}{x}\right) = \text{Arctan}\left(\frac{y}{x}\right). \end{aligned}$$

One may, however doubt, if most readers would introduce the new variable $t = \frac{y}{x}$ for y constant.

◇

INDEFINITE INTEGRATION OF THE LATTER TERM OF (2). When $x > 0$ is considered as a constant, we get

$$F_2(x, y) = \int \frac{x}{x^2 + y^2} dy = \int \frac{1}{1 + \left(\frac{y}{x}\right)^2} d\left(\frac{y}{x}\right) = \text{Arctan}\left(\frac{y}{x}\right).$$

Then

$$dF_2 = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \left(-\frac{y}{x^2}\right) dx + \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} dy = -\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy = \omega - dF_1,$$

hence by a rearrangement,

$$\omega = \mathbf{V} \cdot d\mathbf{x} = dF_1 + dF_2 = d(F_1 + F_2) = d\left\{z^2 \ln(x^2 + y^2) + \text{Arctan}\left(\frac{y}{x}\right)\right\}$$

for $x > 0$.

We conclude that \mathbf{V} is a gradient field in A with its integrals given by

$$F(x, y, z) = z^2 \ln(x^2 + y^2) + \text{Arctan}\left(\frac{y}{x}\right) + k$$

for $k \in \mathbb{R}$ and $(x, y, z) \in A$.

It follows from $F(1, 0, 0) = 0 + 0 + k = 0$ that $k = 0$, thus the searched integral is

$$F(x, y, z) = z^2 \ln(x^2 + y^2) + \operatorname{Arctan}\left(\frac{y}{x}\right), \quad (x, y, z) \in A.$$

Third method. It follows by a tangential line integral along the curve given by the line segments

$$(1, 0, 0) \longrightarrow (x, 0, 0) \longrightarrow (x, y, 0) \longrightarrow (x, y, z)$$

that the curve lies entirely in A , and that the result (the candidate) automatically fulfils $F(1, 0, 0)$. Hence

$$\begin{aligned} F(x, y, z) &= \int_1^x 0 \, dt + \int_0^y \frac{x}{x^2 + t^2} \, dt + \int_0^z 2t \ln(x^2 + y^2) \, dt \\ &= z^2 \ln(x^2 + y^2) + \operatorname{Arctan}\left(\frac{y}{x}\right). \end{aligned}$$

C When this method is used, one *shall always* check the result:

$$\begin{aligned} \nabla F &= \left(\frac{1}{1 + \left(\frac{y}{x}\right)^2} \left(\frac{y}{x^2}\right) + \frac{z^2 \cdot 2x}{x^2 + y^2}, \frac{1}{1 + \left(\frac{y}{x}\right)^2} \frac{1}{x} + \frac{z^2 \cdot 2y}{x^2 + y^2}, 2z \ln(x^2 + y^2) \right) \\ &= \left(-\frac{y}{x^2 + y^2} + \frac{2xz^2}{x^2 + y^2}, \frac{x}{x^2 + y^2} + \frac{2yz^2}{x^2 + y^2}, 2z \ln(x^2 + y^2) \right) \\ &= \left(\frac{2xz^2 - y}{x^2 + y^2}, \frac{2yz^2 + x}{x^2 + y^2}, 2z \ln(x^2 + y^2) \right) = \mathbf{V}(x, y, z). \end{aligned}$$

We see that the result is correct, so \mathbf{V} is a gradient field in A , and the searched integral is

$$F(x, y, z) = z^2 \ln(x^2 + y^2) + \operatorname{Arctan}\left(\frac{y}{x}\right).$$

I Line integral.

First method. The curve \mathcal{K} , given by

$$(x, y, z) = \left(1 + t^2, \frac{t}{\sqrt{2}}\right), \quad t \in [0, 1],$$

lies entirely in A with initial point $(1, 0, 0)$ and the end point $\left(1, 1, \frac{1}{\sqrt{2}}\right)$, and

$$F(x, y, z) = z^2 \ln(x^2 + y^2) + \operatorname{Arctan}\left(\frac{y}{x}\right)$$

is a integral of \mathbf{V} in A . Hence

$$\begin{aligned} \int_{\mathcal{K}} \mathbf{V} \cdot d\mathbf{x} &= F\left(1, 1, \frac{1}{\sqrt{2}}\right) - F(1, 0, 0) = F\left(1, 1, \frac{1}{\sqrt{2}}\right) \\ &= \operatorname{Arctan} 1 + \frac{1}{2} \ln(1 + 1) = \frac{\pi}{4} + \frac{1}{2} \ln 2. \end{aligned}$$

Second method. We get by computing

$$\begin{aligned}
 \int_{\mathcal{C}} \mathbf{V} \cdot d\mathbf{x} &= \int_0^1 \left\{ \frac{2 \cdot 1 \cdot \frac{t^2}{2} - t^2}{1+t^4} \cdot 0 + \frac{2 \cdot t^2 \cdot \frac{t^2}{2} + 1}{1+t^4} \cdot 2t + 2 \cdot \frac{t}{\sqrt{2}} \ln(1+t^4) \cdot \frac{1}{\sqrt{2}} \right\} dt \\
 &= \int_0^1 \{2t + t \ln(1+t^4)\} dt = 1 + \frac{1}{2} \int_0^1 \ln(1+u^2) du \\
 &= 1 + \frac{1}{2} [u \ln(1+u^2)]_0^1 - \frac{1}{2} \int_0^1 \frac{2u^2}{1+u^2} du = 1 + \frac{1}{2} \ln 2 - \int_0^1 \frac{u^2+1-1}{u^2+1} du \\
 &= 1 + \frac{1}{2} \ln 2 - 1 + [\text{Arctan } u]_0^1 = \frac{\pi}{4} + \frac{1}{2} \ln 2.
 \end{aligned}$$

CIRCULATION. Let \mathcal{C} be the circle in the XY -plane described parametrically by

$$(x, y, z) = (\cos t, \sin t, 0), \quad t \in [0, 2\pi].$$

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Then the circulation is given by

$$\int_{\mathcal{C}} \mathbf{V} \cdot d\mathbf{x} = \int_0^{2\pi} \{-\sin t \cdot (-\sin t) + \cos t \cdot \cos t + 0\} dt = \int_0^{2\pi} dt = 2\pi$$

which is different from 0, and \mathbf{V} is *not* a gradient field in its entire domain.

We notice that \mathcal{C} traverses points which do not lie in A .

Example 1.4 Given a vector field in the plane,

$$\mathbf{V}(x, y) = (x^2 + y^2, xy), \quad (x, y) \in \mathbb{R}^2.$$

- 1) Find the tangential line integral of \mathbf{V} along the circle $x^2 + y^2 = 1$ run through in the positive sense of the plane.
- 2) Show that \mathbf{V} is not a gradient field.

A Tangential line integral.

D Follow the guidelines.

I 1) Let \mathcal{K} be the unit circle given by

$$(x, y) = (\cos t, \sin t), \quad t \in [0, 2\pi].$$

Then

$$\int_{\mathcal{K}} \mathbf{V} \cdot d\mathbf{x} = \int_0^{2\pi} \{1 \cdot (-\sin t) + \cos t \cdot \sin t \cdot \cos t\} dt = \left[\cos t - \frac{1}{3} \cos^3 t \right]_0^{2\pi} = 0.$$

2) Since

$$\frac{\partial V_1}{\partial y} = 2y \neq \frac{\partial V_2}{\partial x} = y,$$

the necessary condition is *not* satisfied, and the field is not a gradient field.

ALTERNATIVELY we get by integrating along a broken line from $(0, 0, 0)$ that

$$\Phi(x, y, z) = \int_{\mathcal{K}} \mathbf{V} \cdot d\mathbf{x} = \int_0^x t^2 dt + \int_0^y xt dt = \frac{1}{3} x^3 + \frac{1}{2} xy^2,$$

where

$$\nabla \Phi = \left(x^2 + \frac{y^2}{2}, xy \right) \neq \mathbf{V}(x, y),$$

which shows that \mathbf{V} is not a gradient field.

Example 1.5 Given the vector field

$$\mathbf{V}(x, y, z) = \begin{pmatrix} \exp(-(x + 2y + 3z)^2) \\ 2 \exp(-(x + 2y + 3z)^2) \\ 3 \exp(-(x + 2y + 3z)^2) \end{pmatrix}, \quad (x, y, z) \in \mathbb{R}^3.$$

- 1) Prove that \mathbf{V} is a gradient field.
- 2) Let F be an integral of \mathbf{V} . Prove that F has neither a maximum nor a minimum.

A Gradient field.

D Prove directly that $\mathbf{V} \cdot d\mathbf{x} = dF$. ALTERNATIVELY apply the standard method.

I 1) **First method.** We get by inspection,

$$\begin{aligned} \mathbf{V} \cdot d\mathbf{x} &= \exp(-(x + 2y + 3z)^2) \cdot (dx + 2 dy + 3 dz) \\ &= \exp(-(x + 2y + 3z)^2) d(x + 2y + 3z) \\ &= d \left(\int_0^{x+2y+3z} \exp(-t^2) dt + C \right), \end{aligned}$$

thus \mathbf{V} is a gradient field. An integral is e.g.

$$F(x, y, z) = \int_0^{x+2y+3z} \exp(-t^2) dt.$$

This cannot be expressed by the most elementary functions. We have not yet introduced the *error function*.

Second method. Since \mathbb{R}^3 is simply connected, and \mathbf{V} is of class C^∞ , we shall only show that the “mixed derivatives” agree. We get by computing

$$\begin{aligned} \frac{\partial V_x}{\partial y} &= -4(x + 2y + 3z) \exp(-(x + 2y + 3z)^2), \\ \frac{\partial V_x}{\partial z} &= -6(x + 2y + 3z) \exp(-(x + 2y + 3z)^2), \\ \frac{\partial V_y}{\partial x} &= -4(x + 2y + 3z) \exp(-(x + 2y + 3z)^2), \\ \frac{\partial V_y}{\partial z} &= -12(x + 2y + 3z) \exp(-(x + 2y + 3z)^2), \\ \frac{\partial V_z}{\partial x} &= -6(x + 2y + 3z) \exp(-(x + 2y + 3z)^2). \end{aligned}$$

Hence by comparison,

$$\frac{\partial V_x}{\partial y} = \frac{\partial V_y}{\partial x}, \quad \frac{\partial V_x}{\partial z} = \frac{\partial V_z}{\partial x}, \quad \frac{\partial V_y}{\partial z} = \frac{\partial V_z}{\partial y},$$

which together with that \mathbb{R}^3 is simply connected shows that \mathbf{V} is a gradient field.

- 2) As F is of class C^∞ , the (local) maxima and minima can only be attained at stationary points. Since

$$\nabla F = \mathbf{V} \neq \mathbf{0} \quad \text{for all } (x, y, z) \in \mathbb{R}^3,$$

it follows that F has no stationary points and thus no maxima or minima either.

Example 1.6 Check if the vector field

$$\mathbf{V}(x, y) = \left(-\frac{x^2 y^3}{x^6 + y^6}, \frac{x^3 y^2}{x^6 + y^6} \right), \quad (x, y) \neq (0, 0),$$

is a gradient field.

A Check of a possible gradient field.

D Integrate along the unit circle and show that the value is $\neq 0$.

I We use the following parametric description of the unit circle

$$(x, y) = (\cos t, \sin t), \quad t \in [0, 2\pi].$$

Hence,

$$\begin{aligned} \oint_K \mathbf{V} \cdot d\mathbf{x} &= \int_0^{2\pi} \frac{1}{\cos^6 t + \sin^6 t} (-\cos^2 t \sin^3 t, \cos^3 t \sin^2 t) \cdot (-\sin t, \cos t) dt \\ &= \int_0^{2\pi} \frac{1}{\cos^6 t + \sin^6 t} (\cos^2 t \sin^4 t + \cos^4 t \sin^2 t) dt \\ &= \int_0^{2\pi} \frac{\cos^2 t \cdot \sin^2 t}{\cos^6 t + \sin^6 t} dt > 0. \end{aligned}$$

Since $\oint_K \mathbf{V} \cdot d\mathbf{x} \neq 0$ along a closed curve K , the vector field is not a gradient field in $\mathbb{R}^2 \setminus \{(0, 0)\}$.

REMARK. The underhand dealing is that it can be proved that $\mathbf{V}(x, y)$ is a gradient field in every *simply connected* subdomain of $\mathbb{R}^2 \setminus \{(0, 0)\}$. We shall show this in the special case where $x > 0$. We get by some manipulation using that $x > 0$,

$$\begin{aligned} \omega &= \mathbf{V} \cdot d\mathbf{x} = -\frac{x^2 y^3}{x^6 + y^6} dx + \frac{x^3 y^2}{x^6 + y^6} dy = -\frac{1}{x^4} \cdot \frac{y^3}{1 + \left(\frac{y}{x}\right)^6} dx + \frac{1}{x^3} \cdot \frac{1}{1 + \left(\frac{y}{x}\right)^6} y^2 dy \\ &= \frac{1}{3} \cdot \frac{1}{1 + \left(\frac{y}{x}\right)^6} \left\{ y^3 d\left(\frac{1}{x^3}\right) + \frac{1}{x^3} d(y^3) \right\} \\ &= \frac{1}{3} \cdot \frac{1}{1 + \left\{\left(\frac{y}{x}\right)^3\right\}^2} d\left(\left\{\frac{y}{x}\right\}^3\right) = \frac{1}{3} d\text{Arctan}\left(\left\{\frac{y}{x}\right\}^3\right), \end{aligned}$$

proving that the vector field is a gradient field in the right hand half plane $x > 0$. It is by exploiting this idea possible to show that the vector field is a gradient field in every *simply connected* subset of $\mathbb{R}^2 \setminus \{(0, 0)\}$, without being a gradient field in all of $\mathbb{R}^2 \setminus \{(0, 0)\}$. \diamond

Example 1.7 Given the vector field

$$\mathbf{V}(x, y) = \left(\frac{x^2}{\sqrt{y+x^2}} + \sqrt{y+x^2}, \frac{x}{2\sqrt{y+x^2}} \right).$$

- 1) Prove that \mathbf{V} is a gradient field in $\mathbb{R}_+ \times \mathbb{R}_+$, and find all the integrals of \mathbf{V} .
- 2) Check if \mathbf{V} is also a gradient field in the entire domain in which \mathbf{V} is defined.
- 3) Find the tangential line integral of \mathbf{V} along the line segment from $(0, 1)$ to $(1, 0)$.

A Gradient field.

D Sketch the domain. Then use the standard procedure for examples of this kind.

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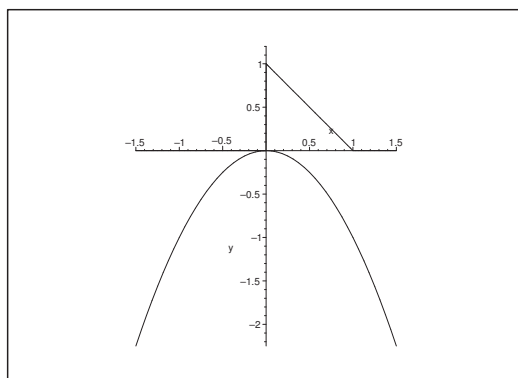


Figure 3: The domain and the curve from $(0, 1)$ to $(1, 0)$.

I The vector field is defined and of class C^∞ , when $y > -x^2$, thus

$$D = \{(x, y) \mid y > -x^2\}.$$

1) Let

$$\omega = \mathbf{V} \cdot d\mathbf{x} = \left(\frac{x^2}{\sqrt{y+x^2}} + \sqrt{y+x^2} \right) dx + \frac{x}{2\sqrt{y+x^2}} dy.$$

Then

$$F(x, y) = \int \frac{x}{2\sqrt{y+x^2}} dy = x\sqrt{y+x^2}$$

and

$$\omega - dF = \omega - \left(\sqrt{y+x^2} + \frac{x^2}{\sqrt{y+x^2}} \right) dx - \frac{x}{2\sqrt{y+x^2}} dy = 0,$$

and the integrals in $\mathbb{R}_+ \times \mathbb{R}_+$ are given by

$$F_C(x, y) = x\sqrt{y+x^2} + C, \quad C \in \mathbb{R}.$$

2) Since $dF_C = \omega$ in all of D , we conclude that \mathbf{V} is a gradient field in D with the same integrals.

3) The line segment between $(0, 1)$ and $(1, 0)$ lies clearly in D . It follows from 2) that

$$\int_{\mathcal{K}} \mathbf{V} \cdot d\mathbf{x} = F(1, 0) - F(0, 1) = \left[x\sqrt{y+x^2} + C \right]_{(0,1)}^{(1,0)} = 1.$$

Example 1.8 Prove that the differential form

$$\omega = (2x + y e^{xy}) dx + (y + x e^{xy}) dy, \quad (x, y) \in \mathbb{R}^2,$$

is exact and find the integral F , for which $F(0, 0) = 0$.

A Exact differential form.

D Use either 1) manipulation, 2) indefinite integration, or 3) integration along a broken line.

I 1) MANIPULATION. When we collect terms which look very similar to each other, we get immediately,

$$\begin{aligned} \omega &= (2x + y e^{xy}) dx + (y + x e^{xy}) dy = (2x dx + y dy) + e^{xy}(y dx + x dy) \\ &= d\left(x^2 + \frac{1}{2}y^2\right) + e^{xy} d(xy) = d\left(x^2 + \frac{1}{2}y^2 + e^{xy}\right), \end{aligned}$$

and the differential form is exact with the integrals

$$F(x, y) = x^2 + \frac{1}{2}y^2 + e^{xy} + C, \quad C \in \mathbb{R}.$$

From $F(0, 0) = 0$ follows that $C = -1$, and the searched solution becomes

$$F(x, y) = x^2 + \frac{1}{2}y^2 + e^{xy} - 1.$$

2) INDEFINITE INTEGRATION. We get for fixed y ,

$$F_1(x, y) = \int (2x + y e^{xy}) dx = x^2 + e^{xy}$$

where

$$\begin{aligned} \omega - dF_1 &= (2x + y e^{xy}) dx + (y + x e^{xy}) dy - (2x + y e^{xy}) dx - x e^{xy} dy \\ &= y dy, \end{aligned}$$

hence $F_2(y) = \int y dy = \frac{1}{2}y^2$, and

$$F(x, y) = F_1(x, y) + F_2(y) + C = x^2 + \frac{1}{2}y^2 + e^{xy} + C, \quad C \in \mathbb{R}.$$

We conclude from $F(0, 0) = 0$ that $C = -1$, so

$$F(x, y) = x^2 + \frac{1}{2}y^2 + e^{xy} - 1.$$

3) INTEGRATION ALONG A BROKEN LINE FROM $(0, 0)$. Here we get

$$F(x, y) = \int_0^x (2t + 0) dt + \int_0^y (t + x e^{xt}) dt = x^2 + \frac{1}{2}y^2 + e^{xy} - 1.$$

Notice that if we apply this method we *must always* check the result:

C TEST. We get by insertion that $F(0, 0) = 0$. Furthermore, we get the differential

$$dF = (2x + y e^{xy}) dx + (y + x e^{xy}) dy = \omega.$$

We see that the differential form is exact.

Example 1.9 Given the vector field

$$\mathbf{V}(x, y) = (x^2 y^6 - y + x, 2x^3 y^5 - x + y), \quad (x, y) \in \mathbb{R}^2.$$

1. Prove that \mathbf{V} is a gradient field, and then find all integrals of \mathbf{V} .

Let \mathcal{K}_1 denote the circle $x^2 + y^2 = 1$ run through once in the positive sense of the plane, and let \mathcal{K}_2 denote the oriented line segment from $(0, 0)$ to $(1, 1)$.

2. Find the tangential line integrals $\int_{\mathcal{K}_1} \mathbf{V} \cdot d\mathbf{x}$ and $\int_{\mathcal{K}_2} \mathbf{V} \cdot d\mathbf{x}$.

A Gradient field.

D Find for the practice of the methods the integrals in as many ways as possible. In the latter question we only use the integration theorem, in which only the initial point and the end point of the curve enter.

I 1) **First variant.** By some simple manipulations using the rules of calculations of a differential we get

$$\begin{aligned} \omega &= \mathbf{V} \cdot d\mathbf{x} = (x^2 y^6 - y + x) dx + (2x^3 y^5 - x + y) dy \\ &= \left\{ \frac{1}{3} y^6 d(x^3) + \frac{1}{3} x^3 d(y^6) \right\} - (y dx + x dy) + \frac{1}{2} d(x^2 + y^2) \\ &= d \left\{ \frac{1}{2} (x^2 + y^2) - xy + \frac{1}{3} x^3 y^6 \right\} = d \left\{ \frac{1}{2} (x - y)^2 + \frac{1}{3} x^3 y^6 \right\}. \end{aligned}$$

This shows that \mathbf{V} is a gradient field, and the integrals are given by

$$F(x, y) = \frac{1}{2} (x - y)^2 + \frac{1}{3} x^3 y^6 + C, \quad C \in \mathbb{R}.$$

Second variant. We get by an integration along a broken line from $(0, 0)$

$$\begin{aligned} F(x, y) &= \int_0^x t dt + \int_0^y (2x^3 t^5 - x + t) dt = \frac{1}{2} x^2 + \frac{1}{3} x^3 y^6 - xy + \frac{1}{2} y^2 \\ &= \frac{1}{2} (x - y)^2 + \frac{1}{3} x^3 y^6. \end{aligned}$$

C We shall *always* check the result by this method. However,

$$dF = (x^2 y^6 + x - y) dx + (2x^3 y^5 - x + y) dy = \mathbf{V} \cdot d\mathbf{x},$$

and it follows that \mathbf{V} is a gradient field for which the integrals are given by

$$F(x, y) = \frac{1}{3} x^3 y^6 + \frac{1}{2} (x - y)^2 + C, \quad C \in \mathbb{R}.$$

Third variant. *Indefinite integration.* Let y be arbitrary. Then

$$F_1(x, y) = \int (x^2 y^6 - y + x) dx = \frac{1}{3} x^3 y^6 - xy + \frac{1}{2} x^2,$$

hence

$$dF_1 = (x^2 y^6 - y + x) dx + (2x^3 y^5 - x) dy,$$

and thus

$$\omega - dF_1 = y dy = d\left(\frac{1}{2} y^2\right).$$

It follows that

$$\omega = dF = d\left(F_1 + \frac{1}{2} y^2\right) = d\left\{\frac{1}{3} x^3 y^6 - xy + \frac{1}{2} x^2 + \frac{1}{2} y^2\right\},$$

and \mathbf{V} is a gradient field with the integrals

$$F(x, y) = \frac{1}{3} x^3 y^6 + \frac{1}{2} (x - y)^2 + C, \quad C \in \mathbb{R}.$$

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2) Since \mathbf{V} is a gradient field and \mathcal{K}_1 is a closed curve, we get

$$\int_{\mathcal{K}_1} \mathbf{V} \cdot d\mathbf{x} = 0.$$

The latter line integral depends also only on the initial and the end point of the curve, so

$$\int_{\mathcal{K}_2} \mathbf{V} \cdot d\mathbf{x} = F(1, 1) - F(0, 0) = \frac{1}{3}.$$

Example 1.10 Let a denote a positive constant. Given the vector field

$$\mathbf{V}(x, y) = \left(\frac{1 + x^2 y^2}{x}, \frac{2 + a x^2 y^2}{3y} \right), \quad (x, y) \in \mathbb{R}_+ \times \mathbb{R}_+.$$

- 1) Prove that \mathbf{V} is a gradient field, if and only if $a = 3$.
- 2) Find for $a = 3$ all integrals of \mathbf{V} .

A Gradient field, integral.

D Check $\omega = \mathbf{V} \cdot d\mathbf{x}$.

I 1) As $x > 0$ and $y > 0$, it follows by reduction,

$$\begin{aligned} \omega &= \mathbf{V} \cdot d\mathbf{x} = \frac{1 + x^2 y^2}{x} dx + \frac{2 + a x^2 y^2}{3y} dy = \frac{1}{x} dx + \frac{2}{3} \frac{1}{y} dy + x y^2 dx + \frac{a}{3} x^2 y dy \\ &= d \ln x + \frac{2}{3} d \ln y + \frac{1}{2} \{y^2 d(x^2) + x^2 d(y^2)\} + \left(\frac{a}{3} - 1\right) x^2 y dy \\ &= d \left\{ \ln x + \frac{2}{3} \ln y + \frac{1}{2} x^2 y^2 \right\} + \frac{1}{3} (a - 3) x^2 y dy. \end{aligned}$$

Clearly, this differential form is exact if and only if $a = 3$.

2) When $a = 3$, it follows directly from the above that the integrals are given by

$$F(x, y) = \ln x + \frac{2}{3} \ln y + \frac{1}{2} x^2 y^2 + C,$$

where $C \in \mathbb{R}$ and $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$.

Example 1.11 Sketch the domain of the vector field

$$\mathbf{V}(x, y) = \left(\frac{y}{\sqrt{1 - xy}}, 1 + \frac{x}{\sqrt{1 - xy}} \right).$$

Then prove that \mathbf{V} is a gradient field and find all integrals of \mathbf{V} .

A Gradient field.

D Sketch the domain; check $\omega = \mathbf{V} \cdot d\mathbf{x}$.

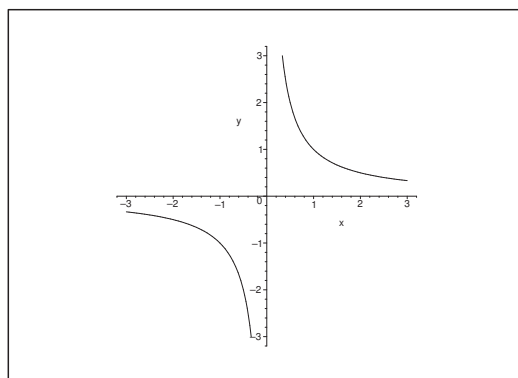


Figure 4: The domain D is the open region lying between the two branches of the hyperbola.

- I** The vector field is defined if and only if $xy < 1$, so D is the open region between the two branches of the hyperbola on the figure.

Then we get by a small rearrangement,

$$\begin{aligned}\omega &= \mathbf{V} \cdot d\mathbf{x} = \frac{y}{\sqrt{1-xy}} dx + \left(1 + \frac{x}{\sqrt{1-xy}}\right) dy \\ &= dy + \frac{1}{\sqrt{1-xy}} (y dx + x dy) = dy + \frac{1}{\sqrt{1-xy}} d(xy) \\ &= dy + d(-2\sqrt{1-xy}) = d\{y - 2\sqrt{1-xy}\}.\end{aligned}$$

Since we can put everything under the d -sign, \mathbf{V} is a gradient field and all integrals in D are given by

$$F(x, y) = y - 2\sqrt{1-xy} + C, \quad C \in \mathbb{R}.$$

Example 1.12 Given the gradient field

$$\mathbf{V}(x, y) = \left(\frac{-y}{(x+1)^2 + y^2}, \frac{x+1}{(x+1)^2 + y^2} \right), \quad y > 0.$$

Find all the integrals of \mathbf{V} .

A Gradient field, integral.

D Check $\omega = \mathbf{V} \cdot d\mathbf{x}$.

I Assuming that $y > 0$ we get by a small manipulation,

$$\begin{aligned}\omega &= \mathbf{V} \cdot d\mathbf{x} = \frac{1}{(x+1)^2 + y^2} (-y dx + (x+1) dy) \\ &= -\frac{1}{1 + \left(\frac{x+1}{y}\right)^2} \left\{ \frac{1}{y} d(x+1) + (x+1) d\left(\frac{1}{y}\right) \right\} \\ &= -\frac{1}{1 + \left(\frac{x+1}{y}\right)^2} d\left(\frac{x+1}{y}\right) = d \operatorname{Arccot} \left(\frac{x+1}{y}\right).\end{aligned}$$

Hence, the integrals are

$$F(x, y) = \operatorname{Arccot} \left(\frac{x+1}{y}\right) + C, \quad C \in \mathbb{R}, \quad y > 0.$$

Example 1.13 Given the vector field

$$\mathbf{V}(x, y) = \left(-\frac{2xy^2}{x^4 + y^4}, \frac{2yx^2}{x^4 + y^4} \right), \quad (x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}.$$

1. Prove that

$$F(x, y) = \operatorname{Arctan} \left(\frac{y^2}{x^2}\right)$$

is an integral of \mathbf{V} in each of the sets

$$A_1 = \{(x, y) \in \mathbb{R}^2 \mid x > 0\} \quad \text{and} \quad A_2 = \{(x, y) \in \mathbb{R}^2 \mid x < 0\}.$$

2. Find all the integrals of \mathbf{V} in each of the sets

$$B_1 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\} \quad \text{and} \quad B_2 = \{(x, y) \in \mathbb{R}^2 \mid y < 0\}.$$

Let \mathcal{K}_1 be the line segment from $(-1, 1)$ to $(1, 1)$, and let \mathcal{K}_2 be the line segment from $(1, 1)$ to $(-1, 1)$.

3. Compute the tangential line integrals

$$\int_{\mathcal{K}_1} \mathbf{V} \cdot \mathbf{t} ds \quad \text{and} \quad \int_{\mathcal{K}_2} \mathbf{V} \cdot \mathbf{t} ds.$$

Let \mathcal{K} be the boundary curve of the square of vertices $(1, -1)$, $(1, 1)$, $(-1, 1)$ and $(-1, -1)$, where \mathcal{K} is oriented in the positive sense of the plane.

4. Find the tangential line integral

$$\int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} ds.$$

A Tangential line integrals. The vector field has an integral in each simply connected domain which does not contain $(0, 0)$. In general one shall take care.

D Follow the guidelines.

I 1) We shall only prove that $\nabla F = \mathbf{V}$ in A_1 and in A_2 . This follows from

$$\nabla F = \left(\frac{-2 \frac{y^2}{x^3}}{1 + \frac{y^4}{x^4}}, \frac{2 \frac{y}{x^2}}{1 + \frac{y^4}{x^4}} \right) = \left(-\frac{2xy^2}{x^4 + y^4}, \frac{2yx^2}{x^4 + y^4} \right) = \mathbf{V}(x, y).$$

2) Clearly, $\text{Arctan}\left(\frac{y^2}{x^2}\right)$ cannot be applied, but if we guess “of symmetric reasons” on

$$F_1(x, y) = -\text{Arctan}\left(\frac{x^2}{y^2}\right),$$

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then we get in B_1 , or in B_2 , respectively, that

$$\nabla F = \begin{pmatrix} -\frac{2x}{y^2} & -\frac{2x^2}{y^3} \\ 1 + \frac{x^4}{y^4} & 1 + \frac{x^4}{y^4} \end{pmatrix} = \begin{pmatrix} -\frac{2xy^2}{x^4 + y^4} & \frac{2yx^2}{x^4 + y^4} \end{pmatrix} = \mathbf{V}(x, y),$$

and the integrals of $\mathbf{V}(x, y)$ in B_1 , or in B_2 , are

$$F_1(x, y) = -\operatorname{Arctan}\left(\frac{x^2}{y^2}\right) + C, \quad C \in \mathbb{R}.$$

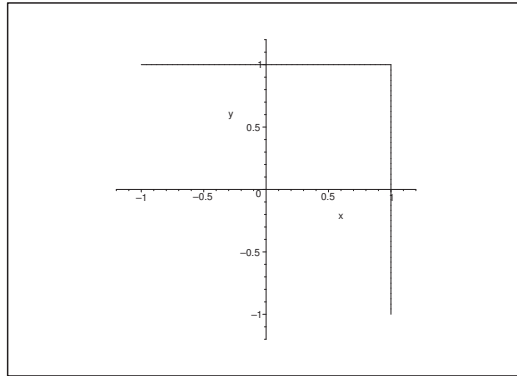


Figure 5: The curve composed of \mathcal{K}_1 from $(1, -1)$ to $(1, 1)$, and \mathcal{K}_2 from $(1, 1)$ to $(-1, 1)$.

- 3) The line segment \mathcal{K}_1 lies entirely in A_1 , and the line segment \mathcal{K}_2 lies entirely in B_1 . Hence we can use the integrals found in A_1 and in B_1 , respectively. Thus

$$\int_{\mathcal{K}_1} \mathbf{V} \cdot \mathbf{t} \, ds = [F(x, y)]_{(1, -1)}^{(1, 1)} = \operatorname{Arctan} 1 - \operatorname{Arctan} 1 = 0,$$

and

$$\int_{\mathcal{K}_2} \mathbf{V} \cdot \mathbf{t} \, ds = [F(x, y)]_{(1, 1)}^{(-1, 1)} = -\operatorname{Arctan} 1 + \operatorname{Arctan} 1 = 0.$$

- 4) Let \mathcal{K}_3 be the line segment from $(-1, 1)$ to $(-1, -1)$. Then \mathcal{K}_3 lies entirely in A_2 , and we get similarly that

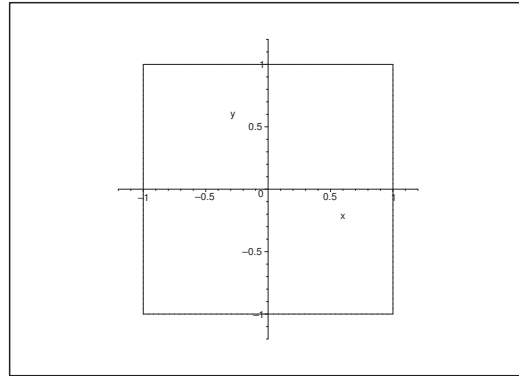
$$\int_{\mathcal{K}_3} \mathbf{V} \cdot \mathbf{t} \, ds = [F(x, y)]_{(-1, 1)}^{(-1, -1)} = 0.$$

If \mathcal{K}_4 is the line segment from $(-1, -1)$ to $(1, -1)$, then \mathcal{K}_4 lies entirely in B_2 , so

$$\int_{\mathcal{K}_4} \mathbf{V} \cdot \mathbf{t} \, ds = [F_1(x, y)]_{(-1, -1)}^{(1, -1)} = 0.$$

Finally, it follows from the results of 3) and 4) that

$$\int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} \, ds = \sum_{j=1}^4 \int_{\mathcal{K}_j} \mathbf{V} \cdot \mathbf{t} \, ds = 0.$$

Figure 6: The curve \mathcal{K} of 4).

REMARK. It is now no longer difficult to show that \mathbf{V} has an integral in the set $\mathbb{R}^2 \setminus \{(0,0)\}$. This shall only be sketched in the following. The idea is to prove that the tangential line integral along any closed (piecewise C^1) curve is 0. If the curve lies in one of the four sets A_1, A_2, B_1, B_2 , this has already been proved. If the closed curve lies in the union of some of these sets without encircling $(0,0)$, just add some curve segments, run through once in each direction, such that each of the closed curves lies in one of the sets A_1, A_2, B_1, B_2 , and the previous result can be applied. Finally, if the closed curve encircles $(0,0)$, we first add the curve $-\mathcal{K}$, i.e. the curve of 4) run through in the opposite direction. Then add some curve segments, also run through once in each direction (contributing with zero to the final line integral) such that each of the new closed curves lies in one of the four sets, in which we have found an integral. The sum of all these closed line integrals is 0, and since we only have added 0 to obtain this result, the original closed line integral must also be 0. And the claim is proved. \diamond

Example 1.14 1) Sketch the domain A of

$$\mathbf{V}(x,y) = \left(\frac{x}{\sqrt{12-3x^2-3y^2}}, \frac{y}{\sqrt{12-3x^2-3y^2}} + \frac{1}{\sqrt{y}} \right).$$

2) Prove that the vector field $\mathbf{V} : A \rightarrow \mathbb{R}^2$ is a gradient field and find all its integrals.

A Domain; gradient field.

D Check where the denominators are defined and $\neq 0$. Then use one of the standard methods of investigation of gradient fields.

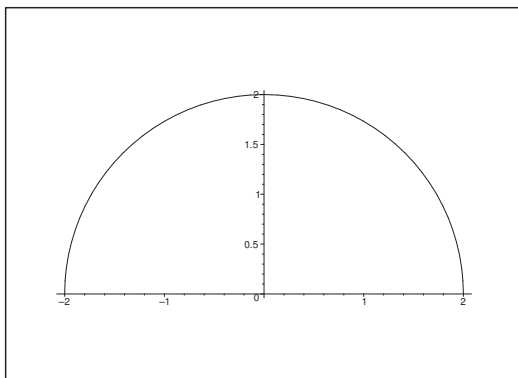
I 1) The domain is given by

$$12 - 3x^2 - 3y^2 > 0 \quad \text{and} \quad y > 0,$$

which we rewrite as

$$x^2 + y^2 < 4 = 2^2 \quad \text{and} \quad y > 0.$$

The domain A is the open half disc of centrum $(0,0)$ and radius 2 and $y > 0$.

Figure 7: The domain A is an open half disc.

2) **First variant.** *Manipulation.* By some small rearrangements,

$$\begin{aligned} \mathbf{V} \cdot d\mathbf{x} &= \frac{xdx + ydy}{\sqrt{12-3x^2-3y^2}} + \frac{dy}{\sqrt{y}} = -\frac{1}{6} \frac{d(12-3x^2-3y^2)}{\sqrt{12-3x^2-3y^2}} + 2d(\sqrt{y}) \\ &= -\frac{1}{3} d\left(\sqrt{12-3x^2-3y^2}\right) + 2d(\sqrt{y}) \\ &= d\left(2\sqrt{y} - \frac{1}{3}\sqrt{12-3x^2-3y^2}\right). \end{aligned}$$

We conclude that \mathbf{V} is a gradient field and all its integrals are given by

$$F(x, y) = 2\sqrt{y} - \frac{1}{3}\sqrt{12-3x^2-3y^2} + C, \quad (x, y) \in A,$$

where C is an arbitrary constant.

Second variant. *Indefinite integration.* When the second coordinate is integrated we get the candidate

$$F_1(x, y) = \int \left(\frac{y}{\sqrt{12-3x^2-3y^2}} + \frac{1}{\sqrt{y}} \right) dy = 2\sqrt{y} - \frac{1}{3}\sqrt{12-3x^2-3y^2}.$$

Since

$$\frac{\partial F_1}{\partial x} = -\frac{1}{3} \cdot \frac{1}{2} \cdot \frac{-6x}{\sqrt{12-3x^2-3y^2}} = \frac{x}{\sqrt{12-3x^2-3y^2}} = V_1,$$

it follows that F_1 is an integral and \mathbf{V} is a gradient field.

The integrals are now given by

$$F(x, y) = 2\sqrt{y} - \frac{1}{3}\sqrt{12-3x^2-3y^2} + C, \quad (x, y) \in A,$$

where C is an arbitrary constant.

Third variant. *Integration along a broken line.* In order to stay inside the set A we integrate along the broken line

$$(0, 1) \longrightarrow (0, y) \longrightarrow (x, y).$$

In this way we obtain the candidate for $(x, y) \in A$

$$\begin{aligned}
 F_2(x, y) &= \int_1^y \left(\frac{t}{\sqrt{12-3t^2}} - \frac{1}{\sqrt{t}} \right) dt + \int_0^x \frac{t}{\sqrt{12-3t^2-3y^2}} dt \\
 &= 2\sqrt{y} - 2 - \frac{1}{3} \sqrt{12-3y^2} + 1 - \frac{1}{3} \sqrt{12-3x^2-3y^2} + \frac{1}{3} \sqrt{12-3y^2} \\
 &= 2\sqrt{y} - \frac{1}{3} \sqrt{12-3x^2-3y^2} - 1.
 \end{aligned}$$

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We shall *always* check our result in this variant, because the formula will give a result no matter if the differential form is exact or not. But

$$\frac{\partial F_2}{\partial x} = -\frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{12-3x^2-3y^2}} \cdot (-6x) = \frac{x}{\sqrt{12-3x^2-3y^2}} = V_x(x, y),$$

and

$$\begin{aligned} \frac{\partial F_2}{\partial y} &= \frac{1}{\sqrt{y}} - \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{12-3x^2-3y^2}} \cdot (-6y) \\ &= \frac{y}{\sqrt{12-3x^2-3y^2}} + \frac{1}{\sqrt{y}} = V_y(x, y), \end{aligned}$$

thus

$$\nabla F_2(x, y) = \mathbf{V}(x, y).$$

Hence we have proved that the candidate is indeed an integral. Finally, we conclude that all integrals are given by

$$F(x, y) = 2\sqrt{y} - \frac{1}{3} \sqrt{12-3x^2-3y^2} + C, \quad (x, y) \in A,$$

where C is an arbitrary constant.

2 The flux of a vector field

Example 2.1 Find in each of the following cases the flux of the given vector field through the described oriented surface \mathcal{F} .

- 1) The flux of $\mathbf{V}(x, y, z) = (z, x, -3y^2z)$ through the surface \mathcal{F} given by $x^2 + y^2 = 16$ for $x \geq 0$, $y \geq 0$ and $z \in [0, 5]$, where the normal vector \mathbf{n} is pointing away from the Z -axis.
- 2) The flux of $\mathbf{V}(x, y, z) = (\cos x, 0, \cos x + \cos y)$ through the surface \mathcal{F} given by $(x, y) \in [0, \pi] \times [0, \frac{\pi}{2}]$ and $z = 0$, and where $\mathbf{n} = \mathbf{e}_z$.
- 3) The flux of $\mathbf{V}(x, y, z) = (xy, z^2, 2yz)$ through the surface \mathcal{F} given by $x^2 + y^2 + z^2 = a^2$, and $x \geq 0$, $y \geq 0$, $z \geq 0$, and where \mathbf{n} is pointing away from origo.
- 4) The flux of $\mathbf{V}(x, y, z) = (x + y, x - y, y^2 + z)$ through the surface \mathcal{F} given by $x^2 + y^2 \leq 1$ and $z = xy$, and where $\mathbf{n} \cdot \mathbf{e}_z > 0$.
- 5) The flux of

$$\mathbf{V}(x, y, z) = \frac{1}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} (x, y, z),$$

through the surface \mathcal{F} given by $\varrho \leq a$ and $z = h$, and where $\mathbf{n} = \mathbf{e}_z$.

[Cf. **Example 4.3**].

- 6) The flux of

$$\mathbf{V}(x, y, z) = \frac{1}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} (x, y, z),$$

through the surface \mathcal{F} given by $\varrho = a$ and $z \in [-h, h]$, and where \mathbf{n} is pointing away from the Z -axis.

[Cf. **Example 4.3**].

- 7) The flux of $\mathbf{V}(x, y, z) = (y, x, x + y + z)$ through the surface \mathcal{F} given by the parametric description

$$\mathbf{r}(u, v) = (u \cos v, u \sin v, hv), \quad u \in [0, 1], \quad v \in [0, 2\pi].$$
- 8) The flux of $\mathbf{V}(x, y, z) = (y, -x, z^2)$ through the surface \mathcal{F} given by the parametric description

$$\mathbf{r}(u, v) = \left(\sqrt{u} \cos v, \sqrt{u} \sin v, v^{\frac{3}{2}} \right), \quad 1 \leq u \leq 2, \quad 0 \leq v \leq u.$$
- 9) The flux of $\mathbf{V}(x, y, z) = (yz, -xz, hz)$ through the surface \mathcal{F} given by the parametric description

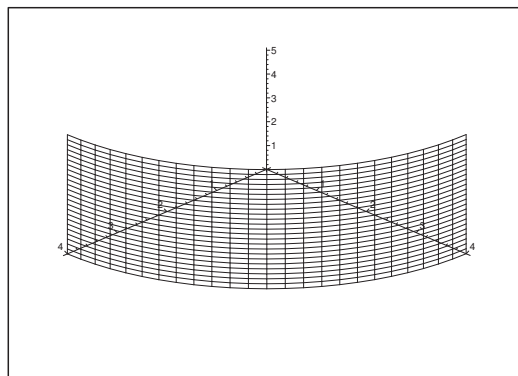
$$\mathbf{r}(u, v) = (u \cos v, u \sin v, hv), \quad u \in [0, 1], \quad v \in [0, 2\pi].$$

A Flux of a vector field through a surface.

D Sketch whenever possible the surface. If the surface is only described in words, set up a parametric description. Compute the normal vector \mathbf{N} (possibly the normed normal vector \mathbf{n}) and check the orientation. Finally, find the flux.

I 1) The surface is in semi polar coordinates described by

$$\varrho = a, \quad \varphi \in \left[0, \frac{\pi}{2}\right], \quad z \in [0, 5],$$

Figure 8: The surface \mathcal{F} of Example 2.1.1.

and the surface is a cylinder with the parameter domain

$$E = \left\{ (\varphi, z) \mid \varphi \in \left[0, \frac{\pi}{2}\right], z \in [0, 5] \right\}.$$

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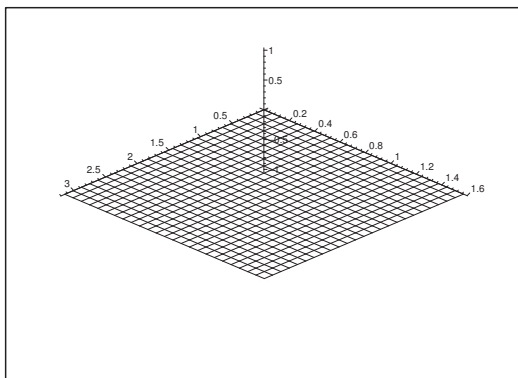
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Figure 9: The surface \mathcal{F} of **Example 2.1.2**.

The unit normal vector is

$$\mathbf{n} = (\cos \varphi, \sin \varphi, 0),$$

and the area element is

$$dS = ds dz = 4 \delta\varphi dz.$$

Hence we get the flux

$$\begin{aligned} \int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n} dS &= \int_E \{z \cos \varphi + 4 \cos \varphi \cdot \sin \varphi\} \cdot 4 d\varphi dz \\ &= 4 \int_0^{\frac{\pi}{2}} \left\{ \int_0^5 (z \cos \varphi + 4 \sin \varphi \cdot \cos \varphi) dz \right\} d\varphi \\ &= 4 \int_0^{\frac{\pi}{2}} \left\{ \frac{25}{2} \cos \varphi + 20 \sin \varphi \cos \varphi \right\} d\varphi = 4 \cdot \frac{25}{2} + 4 \cdot 20 \cdot \frac{1}{2} \\ &= 90. \end{aligned}$$

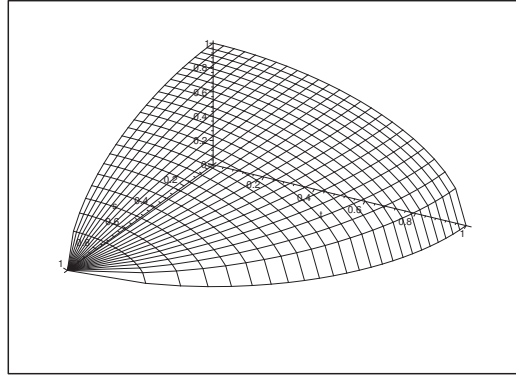
2) In this case the flux is

$$\begin{aligned} \int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n} dS &= \int_0^{\pi} \left\{ \int_0^{\frac{\pi}{2}} (\cos x + \cos y) dy \right\} dx \\ &= \int_0^{\pi} \left\{ \frac{\pi}{2} \cos x + 1 \right\} dx = 0 + 1 \cdot \pi = \pi. \end{aligned}$$

3) The surface is a subset of the sphere of centum $(0, 0, 0)$ and radius a , lying in the first octant.

In *rectangular coordinates* we find the area element on \mathcal{F} ,

$$dS = \frac{a}{\sqrt{a^2 - x^2 - y^2}} dx dy \quad \left(= \frac{a}{z} dx dy \right),$$

Figure 10: The surface \mathcal{F} of **Example 2.1.3** for $a = 1$.

and the unit normal vector is

$$\mathbf{n} = \frac{1}{a}(x, y, z) = \frac{1}{a}(x, y, \sqrt{a^2 - x^2 - y^2}), \quad (x, y) \in E,$$

where the parameter domain is

$$E = \{(x, y) \mid 0 \leq x \leq a, 0 \leq y \leq \sqrt{a^2 - x^2}\}.$$

Then the flux of the vector field $\mathbf{V}(x, y, z) = (xy, z^2, 2yz)$ through \mathcal{F} is

$$\begin{aligned} \int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n} \, dS &= \int_{\mathcal{F}} (xy, z^2, 2yz) \cdot \frac{1}{a}(x, y, z) \, dS \\ &= \frac{1}{a} \int_{\mathcal{F}} \{x^2y + yz^2 + 2yz^2\} \, dS = \frac{1}{a} \int_{\mathcal{F}} y(x^2 + 3z^2) \, dS \\ &= \frac{1}{a} \int_E a \left\{ \frac{yx^2}{\sqrt{a^2 - x^2 - y^2}} + 3y\sqrt{a^2 - x^2 - y^2} \right\} dx \, dy \\ &= \int_0^a \left\{ \int_0^{\sqrt{a^2 - x^2}} \left\{ \frac{x^2}{\sqrt{a^2 - x^2 - y^2}} + 3\sqrt{a^2 - x^2 - y^2} \right\} y \, dy \right\} dx \\ &= \frac{1}{2} \int_0^a \left\{ \int_0^{a^2 - x^2} \left(\frac{x^2}{\sqrt{a^2 - x^2 - t^2}} + 3\sqrt{a^2 - x^2 - t^2} \right) dt \right\} dx \\ &= \frac{1}{2} \int_0^a \left[-2x^2 \sqrt{a^2 - x^2 - t^2} - 3 \cdot \frac{2}{3} \left(\sqrt{a^2 - x^2 - t^2} \right)^3 \right]_{t=0}^{a^2 - x^2} dx \\ &= \int_0^a \left\{ x^2 \sqrt{a^2 - x^2} + (a^2 - x^2) \sqrt{a^2 - x^2} \right\} dx \\ &= a^2 \int_0^a \sqrt{a^2 - x^2} \, dx = a^2 \cdot \frac{\pi}{4} \cdot a^2 = \frac{\pi a^4}{4}. \end{aligned}$$

ALTERNATIVELY, the area element on \mathcal{F} is given in *polar coordinates* by

$$dS = a^2 \sin \theta \, d\theta \, d\varphi, \quad \theta \in \left[0, \frac{\pi}{2}\right], \quad \varphi \in \left[0, \frac{\pi}{2}\right],$$

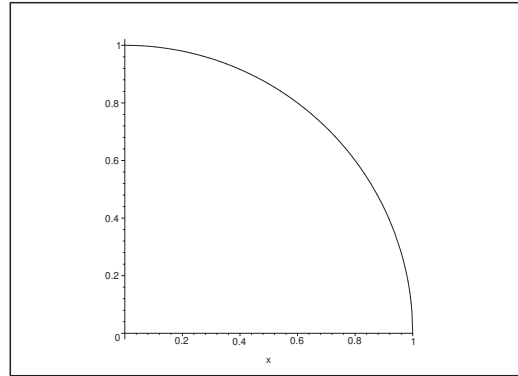


Figure 11: The parameter domain of **Example 2.1.3** for $a = 1$.

thus the parameter domain is

$$E = \left\{ (\theta, \varphi) \mid 0 \leq \theta \leq \frac{\pi}{2}, 0 \leq \varphi \leq \frac{\pi}{2} \right\} = \left[0, \frac{\pi}{2} \right] \times \left[0, \frac{\pi}{2} \right].$$

As

$$(x, y, z) = a (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta),$$

the unit normal vector is

$$\mathbf{n} = \frac{1}{a} (x, y, z) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta).$$

The flux of the vector field

$$\mathbf{V}(x, y, z) = (xy, z^2, 2yz)$$

through the surface \mathcal{F} is

$$\begin{aligned} \int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n} \, dS &= \int_{\mathcal{F}} (xy, z^2, 2yz) \cdot \frac{1}{a} (x, y, z) \, dS \\ &= \frac{1}{2} \int_{\mathcal{F}} \{x^2y + yz^2 + 2yz^2\} dS = \frac{1}{a} \int_{\mathcal{F}} y(x^2 + 3z^2) \, dS \\ &= \frac{1}{a} \int_E a \sin \theta \sin \varphi \cdot a^2 \{\sin^2 \theta \cos^2 \varphi + 3 \cos^2 \theta\} \cdot a^2 \sin \theta \, d\theta \, d\varphi \\ &= a^4 \int_0^{\frac{\pi}{2}} \left\{ \int_0^{\frac{\pi}{2}} \sin^2 \theta (\sin^2 \theta \cos^2 \varphi + 3 \cos^2 \theta) \sin \varphi \, d\varphi \right\} d\theta \\ &= a^4 \int_0^{\frac{\pi}{2}} \sin^2 \theta \left[-\frac{1}{3} \sin^2 \theta \cos^3 \varphi - \frac{1}{3} \cos^2 \theta \cos \varphi \right]_{\varphi=0}^{\frac{\pi}{2}} d\theta \\ &= a^4 \int_0^{\frac{\pi}{2}} \sin^2 \theta \left(\frac{1}{3} \sin^2 \theta + 3 \cos 2\theta \right) d\theta. \end{aligned}$$

We compute the integrand by introducing the double angle,

$$\begin{aligned} & \int^{\theta} (3 \cos^2 \theta + \frac{1}{3} \sin^2 \theta) d\theta \\ &= \frac{1}{2} (1 - \cos 2\theta) \left\{ \frac{3}{2} (1 + \cos 2\theta) + \frac{1}{6} (1 - \cos 2\theta) \right\} \\ &= \frac{1}{12} (1 - \cos 2\theta) \{9(1 + \cos 2\theta) + (1 - \cos 2\theta)\} \\ &= \frac{1}{12} (1 - \cos 2\theta) (10 + 8 \cos 2\theta) = \frac{1}{6} (1 - \cos 2\theta) (5 + 4 \cos 2\theta) \\ &= \frac{1}{6} (5 - \cos 2\theta - 4 \cos^2 2\theta) = \frac{1}{6} \{5 - \cos 2\theta - 2(1 + \cos 4\theta)\} \\ &= \frac{1}{6} (3 - \cos 2\theta - 2 \cos 4\theta) = \frac{1}{2} - \frac{1}{6} \cos 2\theta - \frac{1}{3} \cos 4\theta. \end{aligned}$$

The flux is obtained by insertion,

$$\begin{aligned} \int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n} dS &= a^4 \int_0^{\frac{\pi}{2}} \sin^2 \theta \left(\frac{1}{3} \sin^2 \theta + 3 \cos^2 \theta \right) d\theta \\ &= a^4 \int_0^{\frac{\pi}{2}} \left\{ \frac{1}{2} - \frac{1}{6} \cos 2\theta - \frac{1}{3} \cos 4\theta \right\} d\theta \\ &= a^4 \cdot \frac{1}{2} \cdot \frac{\pi}{2} - a^4 \cdot \frac{1}{6} \cdot \frac{1}{2} [\sin 2\theta]_0^{\frac{\pi}{2}} - a^4 \cdot 13 \cdot \frac{1}{4} [\sin 4\theta]_0^{\frac{\pi}{2}} = \frac{\pi a^4}{4}. \end{aligned}$$

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- 4) Let $E = \{(x, y) \mid x^2 + y^2 \leq 1\}$ be the unit disc. Then a parametric description of the surface \mathcal{F} is given by

$$\{(x, y, xy) \mid (x, y) \in E\},$$

where the normal vector is

$$\mathbf{N}(x, y) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 1 & 0 & y \\ 0 & 1 & x \end{vmatrix} = (-y, -x, 1),$$

and clearly, $\mathbf{N} \cdot \mathbf{e}_z = 1 > 0$.

Then the flux of the vector field

$$\mathbf{V}(x, y, z) = (x + y, x - y, y^2 + z)$$

through \mathcal{F} is given by

$$\begin{aligned} \int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n} \, dS &= \int_E \mathbf{V} \cdot \mathbf{N} \, dx \, dy = \int_E (x + y, x - y, y^2 + xy) \cdot (-y, -x, 1) \, dx \, dy \\ &= \int_E \{-xy - y^2 - x^2 + xy + y^2 + xy\} \, dx \, dy = \int_E (xy - x^2) \, dx \, dy \\ &= \int_0^{2\pi} \left\{ \int_0^1 \varrho^2 (\cos \varphi \cdot \sin \varphi - \cos^2 \varphi) \varrho \, d\varrho \right\} d\varphi \\ &= \frac{1}{4} \int_0^{2\pi} (\cos \varphi \cdot \sin \varphi - \cos^2 \varphi) \, d\varphi \\ &= 0 - \frac{1}{4} \cdot 2\pi \cdot \frac{1}{2} = -\frac{\pi}{4}. \end{aligned}$$

- 5) The surface \mathcal{F} is a disc parallel to the XY -plane at the height h . We choose

$$E = \{(x, y) \mid x^2 + y^2 = \varrho^2 \leq a^2\},$$

as the parameter domain. Then the flux through \mathcal{F} is

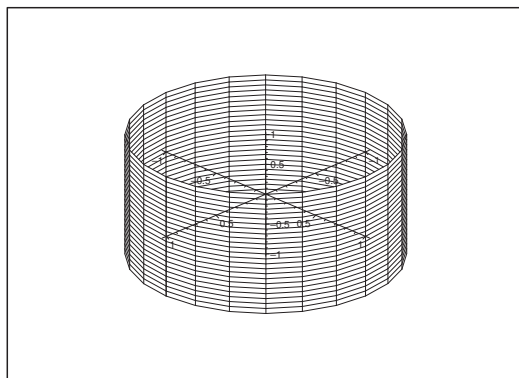
$$\begin{aligned} \int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n} \, dS &= \int_E \frac{h}{(x^2 + y^2 + h^2)^{\frac{3}{2}}} \, dx \, dy = h \int_0^{2\pi} \left\{ \int_0^a \frac{1}{(\varrho^2 + h^2)^{\frac{3}{2}}} \varrho \, d\varrho \right\} d\varphi \\ &= h \cdot 2\pi \left[\frac{1}{2} (-2) \frac{1}{\sqrt{\varrho^2 + h^2}} \right]_{\varrho=0}^a = 2\pi h \left(\frac{1}{\sqrt{h^2}} - \frac{1}{\sqrt{a^2 + h^2}} \right) \\ &= 2\pi \left(1 - \frac{h}{\sqrt{a^2 + h^2}} \right). \end{aligned}$$

- 6) In this case \mathcal{F} is a cylinder surface which is given in semi polar coordinates by the parametric description

$$\{(a, \varphi, z) \mid \varphi \in [0, 2\pi], z \in [-h, h]\},$$

and the parameter domain becomes

$$E = \{(\varphi, z) \mid \varphi \in [0, 2\pi], z \in [-h, h]\} = [0, 2\pi] \times [-h, h].$$

Figure 12: The surface \mathcal{F} of **Example 2.1.6** for $a = 1$ og $h = 1$.

The unit normal vector pointing away from the Z -axis is

$$\mathbf{n} = (\cos \varphi, \sin \varphi, 0),$$

and the area element on \mathcal{F} is

$$dS = ds dz = a d\varphi dz,$$

thus the flux through \mathcal{F} is

$$\begin{aligned} \int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n} dD &= \int_E \frac{a}{(a^2 + z^2)^{\frac{3}{2}}} (\cos^2 \varphi + \sin^2 \varphi + 0) a d\varphi dz \\ &= a^2 \cdot 2\pi \int_{-h}^h \frac{1}{(a^2 + z^2)^{\frac{3}{2}}} dz = 4\pi a^2 \int_0^h \frac{1}{(a^2 + z^2)^{\frac{3}{2}}} dz. \end{aligned}$$

It is natural here to introduce the substitution

$$z = a \sinh t, \quad dz = a \cosh t dt, \quad t = \operatorname{Arsinh} \left(\frac{z}{a} \right).$$

Then we get the flux through the surface

$$\begin{aligned} \int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n} dS &= 4\pi a^2 \int_0^{\operatorname{Arsinh}(\frac{h}{a})} \frac{a \cosh t}{a^3 \cosh^3 t} dt = 4\pi \int_0^{\operatorname{Arsinh}(\frac{h}{a})} \frac{dt}{\cosh^2 t} \\ &= 4\pi [\tanh t]_0^{\operatorname{Arsinh}(\frac{h}{a})} = 4\pi \left[\frac{\sinh t}{\sqrt{1 + \sinh^2 t}} \right]_0^{\operatorname{Arsinh}(\frac{h}{a})} = 4\pi \cdot \frac{\frac{h}{a}}{\sqrt{1 + \frac{h^2}{a^2}}} \\ &= \frac{4\pi h}{\sqrt{a^2 + h^2}}. \end{aligned}$$

REMARK. The field of **Example 2.1.5** and **Example 2.1.6** is the so-called *Coulomb field*, cf. **Example 3.3**. It is tempting to combine the results of **Example 2.1.5** and **Example 2.1.6** to find the flux of the Coulomb field through the surface of the whole cylinder. Since $\mathbf{n} = -\mathbf{e}_z$, when we consider the surface of **Example 2.1.5** at height $-h$, it follows that

$$\text{flux} = 2\pi \left(1 - \frac{h}{\sqrt{a^2 + h^2}} \right) + \frac{4\pi h}{\sqrt{a^2 + h^2}} - 2\pi \left(\frac{-h}{\sqrt{h^2}} - \frac{(-h)}{\sqrt{a^2 + h^2}} \right) = 4\pi. \quad \diamond$$

7) Here

$$\mathbf{N}(u, v) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & h \end{vmatrix} = (h \sin v, -h \cos v, u),$$

so the flux of the vector field $(y, x, x + y + z)$ through \mathcal{F} is

$$\begin{aligned} \int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n} \, dS &= \int_E \mathbf{V} \cdot \mathbf{N}(u, v) \, du \, dv \\ &= \int_E (u \sin v, u \cos v, u(\cos v + \sin v) + hv) \cdot (h \sin v, -h \cos v, u) \, du \, dv \\ &= \int_E (hu \sin^2 v - hu \cos^2 v + u^2(\cos v + \sin v) + huv) \, du \, dv \\ &= \int_E hu(-\cos 2v) \, du \, dv + \int_E u^2(\cos v + \sin v) \, du \, dv + h \int_E uv \, du \, dv \\ &= 0 + 0 + h \int_0^1 u \, du \int_0^{2\pi} v \, dv = h \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot 4\pi^2 = h\pi^2. \end{aligned}$$

8) The normal vector of the surface \mathcal{F} of the parametric description

$$\mathbf{r}(u, v) = (\sqrt{u} \cos v, \sqrt{u} \sin v, v^{3/2}), \quad 1 \leq u \leq 2, \quad 0 \leq v \leq u,$$

is

$$\begin{aligned} \mathbf{N}(u, v) &= \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{1}{2\sqrt{u}} \cos v & \frac{1}{2\sqrt{u}} \sin v & 0 \\ -\sqrt{u} \sin v & \sqrt{u} \cos v & \frac{3}{2} \sqrt{v} \end{vmatrix} \\ &= \left(\frac{3}{4} \sqrt{\frac{v}{u}} \sin v, -\frac{3}{4} \sqrt{\frac{v}{u}} \cos v, \frac{1}{2} \right). \end{aligned}$$

The flux of $\mathbf{V}(x, y, z) = (y, -x, z^2)$ through \mathcal{F} is

$$\begin{aligned} \int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n} \, dS &= \int_E \mathbf{V}(u, v) \cdot \mathbf{N}(u, v) \, du \, dv \\ &= \int_E (\sqrt{u} \sin v, -\sqrt{u} \cos v, v^3) \cdot \left(\frac{3}{4} \sqrt{\frac{v}{u}} \sin v, -\frac{3}{4} \sqrt{\frac{v}{u}} \cos v, \frac{1}{2} \right) \, du \, dv \\ &= \int_E \left\{ \frac{3}{4} \sqrt{v} \sin^2 v + \frac{3}{4} \sqrt{v} \cos^2 v + \frac{1}{2} v^3 \right\} \, du \, dv \\ &= \int_E \left\{ \frac{3}{4} \sqrt{v} + \frac{1}{2} v^3 \right\} \, du \, dv = \int_1^2 \left\{ \int_0^u \left(\frac{3}{4} v^{\frac{1}{2}} + \frac{1}{2} v^3 \right) \, dv \right\} \, du \\ &= \int_1^2 \left[\frac{3}{4} \cdot \frac{2}{3} v^{\frac{3}{2}} + \frac{1}{8} v^4 \right]_0^u \, du = \int_1^2 \left(\frac{1}{2} u^{\frac{3}{2}} + \frac{1}{8} u^4 \right) \, du \\ &= \left[\frac{1}{2} \cdot \frac{2}{5} u^{\frac{5}{2}} + \frac{1}{40} u^5 \right]_1^2 = \frac{1}{5} (\sqrt{2})^5 + \frac{1}{40} \cdot 2^5 - \frac{1}{5} - \frac{1}{40} \\ &= \frac{1}{40} (8 \cdot 4\sqrt{2} + 32 - 8 - 1) = \frac{1}{40} (32\sqrt{2} + 23) = \frac{4\sqrt{2}}{5} + \frac{23}{40}. \end{aligned}$$

9) Here we have [cf. **Example 2.1.7**]

$$\mathbf{N}(u, v) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & h \end{vmatrix} = (h \sin v, -h \cos v, u),$$

and the flux of the vector field $(yz, -xz, hz)$ through the surface \mathcal{F} becomes

$$\begin{aligned} \int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n} \, dS &= \int_E \mathbf{V} \cdot \mathbf{N}(u, v) \, du \, dv \\ &= \int_E (uhv \sin v, -uhv \cos v, h^2v) \cdot (h \sin v, -h \cos v, u) \, du \, dv \\ &= h \int_E (uh \sin^2 v + uh \cos^2 v + huv) \, du \, dv = h^2 \int_E u(1 + v) \, du \, dv \\ &= h^2 \int_0^1 u \, du \cdot \int_0^{2\pi} (v + 1) \, dv = h^2 \cdot \frac{1}{2} \left[\frac{v^2}{2} + v \right]_0^{2\pi} = \frac{h^2}{2} \cdot \{2\pi^2 + 2\pi\} = h^2\pi(\pi + 1). \end{aligned}$$

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3 The divergence and rotation of a vector field

Example 3.1 Find coordinate expressions of the vector fields $\text{rot}(\text{rot } \mathbf{V})$ and $\text{grad}(\text{div } \mathbf{V})$.

Then prove the formula

$$\text{grad}(\text{div } \mathbf{V}) - \text{rot}(\text{rot } \mathbf{V}) = (\text{div}(\text{grad } V_x), \text{div}(\text{grad } V_y), \text{div}(\text{grad } V_z)).$$

A Calculation with nabla.

D The results can be obtained by very mechanical calculations. It is a matter of taste whether one prefers the notation above or

$$\text{grad} = \nabla, \quad \text{div} = \nabla \cdot, \quad \text{rot} = \nabla \times .$$

We shall here use the latter, thereby keeping the formal connection to the geometric relationships that the operations are describing.

I Let \mathbf{V} be a vector field of class C^2 . Then

$$\text{rot } \mathbf{V} = \nabla \times \mathbf{V} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) \mathbf{e}_x + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right) \mathbf{e}_y + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) \mathbf{e}_z.$$

By repeating this pattern we get for the double rotation that

$$\begin{aligned} \text{rot}(\text{rot } \mathbf{V}) &= \nabla \times (\nabla \times \mathbf{V}) \\ &= \left(\frac{\partial}{\partial y} \left\{ \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right\} - \frac{\partial}{\partial z} \left\{ \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right\} \right) \mathbf{e}_x \\ &\quad + \left(\frac{\partial}{\partial z} \left\{ \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right\} - \frac{\partial}{\partial x} \left\{ \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right\} \right) \mathbf{e}_y \\ &\quad + \left(\frac{\partial}{\partial x} \left\{ \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right\} - \frac{\partial}{\partial y} \left\{ \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right\} \right) \mathbf{e}_z, \end{aligned}$$

thus

$$\begin{aligned} \text{rot}(\text{rot } \mathbf{V}) &= \nabla \times (\nabla \times \mathbf{V}) \\ &= \left(-\frac{\partial^2 V_x}{\partial y^2} - \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial^2 V_z}{\partial x \partial z} + \frac{\partial^2 V_y}{\partial x \partial y} \right) \mathbf{e}_x \\ &\quad + \left(-\frac{\partial^2 V_y}{\partial x^2} - \frac{\partial^2 V_y}{\partial z^2} + \frac{\partial^2 V_z}{\partial z^2} + \frac{\partial^2 V_x}{\partial x \partial y} + \frac{\partial^2 V_z}{\partial y \partial z} \right) \mathbf{e}_y \\ &\quad + \left(-\frac{\partial^2 V_z}{\partial x^2} - \frac{\partial^2 V_z}{\partial y^2} + \frac{\partial^2 V_y}{\partial y \partial z} + \frac{\partial^2 V_x}{\partial x \partial z} \right) \mathbf{e}_z \\ &= -(\nabla^2 V_x, \nabla^2 V_y, \nabla^2 V_z) \\ &\quad + \frac{\partial}{\partial x} \left\{ \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \right\} \mathbf{e}_x \\ &\quad + \frac{\partial}{\partial y} \left\{ \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \right\} \mathbf{e}_y \\ &\quad + \frac{\partial}{\partial z} \left\{ \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \right\} \mathbf{e}_z \\ &= -\nabla^2 \mathbf{V} + \nabla(\nabla \cdot \mathbf{V}) \\ &= -(\text{div}(\text{grad } V_x), \text{div}(\text{grad } V_y), \text{div}(\text{grad } V_z)) + \text{grad}(\text{div } \mathbf{V}), \end{aligned}$$

and the formula follows by a rearrangement.

REMARK 1. Notice that the formula can also be written

$$\nabla(\nabla \cdot \mathbf{V}) - \nabla \times (\nabla \times \mathbf{V}) = \nabla^2 \mathbf{V}. \quad \diamond$$

REMARK 2. We note for completeness

$$\begin{aligned} \nabla(\nabla \cdot \mathbf{V}) &= \nabla \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \right) \\ &= \left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_y}{\partial x \partial y} + \frac{\partial^2 V_z}{\partial x \partial z} \right) \mathbf{e}_x + \left(\frac{\partial^2 V_x}{\partial x \partial y} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_z}{\partial y \partial z} \right) \mathbf{e}_y \\ &\quad + \left(\frac{\partial^2 V_x}{\partial x \partial z} + \frac{\partial^2 V_y}{\partial y \partial z} + \frac{\partial^2 V_z}{\partial z^2} \right) \mathbf{e}_z. \quad \diamond \end{aligned}$$

Example 3.2 Find $\operatorname{div} \mathbf{V}$ and $\operatorname{rot} \mathbf{V}$ for each of the following vector fields on \mathbb{R}^3 .

- 1) $\mathbf{V}(x, y, z) = (xz, -y^2, 2x^2y)$.
- 2) $\mathbf{V}(x, y, z) = (z + \sin y, -z + \cos y, 0)$.
- 3) $\mathbf{V}(x, y, z) = (e^{xy}, \cos(xy), \cos(xz^2))$.
- 4) $\mathbf{V}(x, y, z) = (x^2 + yz, y^2 + xz, z^2 + xy)$.
- 5) $\mathbf{V}(x, y, z) = (x + \operatorname{Arctan} y, 3x - z, 2y^z)$.
- 6) $\mathbf{V}(x, y, z) = (xz^3, -2x^2yz, 2yz^4)$.
- 7) $\mathbf{V}(x, y, z) = (\sinh(xyz), z, x)$.
- 8) $\mathbf{V}(\operatorname{Arctan} z, \operatorname{Arctan} x, \operatorname{Arctan} y)$.

A This is just a simple exercise in finding the divergence and the rotation.

D Insert into the formulæ

$$\operatorname{div} \mathbf{V} = \nabla \cdot \mathbf{V} \quad \text{og} \quad \operatorname{rot} \mathbf{V} = \nabla \times \mathbf{V}.$$

I 1) We get for $\mathbf{V} = (xz, -y^2, 2x^2y)$,

$$\operatorname{div} \mathbf{V} = z - 2y + 0 = z - 2y$$

and

$$\operatorname{rot} \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & -y^2 & 2x^2y \end{vmatrix} = (2x^2, x - 4xy, 0).$$

2) We get for $\mathbf{V} = (z + \sin y, -z + \cos y, 0)$,

$$\operatorname{div} \mathbf{V} = 0 - \sin y + 0 = -\sin y$$

and

$$\operatorname{rot} \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z + \sin y & -z + \cos y & 0 \end{vmatrix} = (1, 1, -\cos y).$$

3) We get for $\mathbf{V} = (e^{xy}, \cos(xy), \cos(xz^2))$,

$$\operatorname{div} \mathbf{V} = y e^{xy} - x \sin(xy) - 2xz \sin(xz^2)$$

and

$$\operatorname{rot} \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{xy} & \cos(xy) & \cos(xz^2) \end{vmatrix} = (0, z^2 \sin(xz^2), -y \sin(xy) - x e^{xy}).$$

4) We get for $\mathbf{V} = (x^2 + yz, y^2 + xz, z^2 + xy)$ fås,

$$\operatorname{div} \mathbf{V} = 2(x + y + z)$$

and

$$\operatorname{rot} \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + yz & y^2 + xz & z^2 + xy \end{vmatrix} = (x - x, y - y, z - z) = (0, 0, 0) = \mathbf{0}.$$

5) When

$$\mathbf{V} = (x + \operatorname{Arctan} y, 3x - z, e^{\ln 2 \cdot yz}) = (x + \operatorname{Arctan} y, 3x - z, e^{\ln 2 \cdot yz})$$

we find

$$\operatorname{div} \mathbf{V} = 1 + 0 + \ln 2 \cdot y \cdot e^{\ln 2 \cdot yz} = 1 + \ln 2 \cdot y \cdot 2^{yz}$$

and

$$\operatorname{rot} \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + \operatorname{Arctan} y & 3x - z & e^{\ln 2 \cdot yz} \end{vmatrix} = \left(\ln 2 \cdot z \cdot 2^{yz} + 1, 0, 3 - \frac{1}{1 + y^2} \right).$$

6) If $\mathbf{V} = (xz^3, -2x^2yz, 2yz^4)$, then

$$\operatorname{div} \mathbf{V} = z^3 - 2x^2z + 8yz^3$$

and

$$\operatorname{rot} \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix} = (2z^4 + 2x^2y, 3xz^2, -4xyz).$$

7) If $\mathbf{V} = (\sinh(xyz), z, x)$, then

$$\operatorname{div} \mathbf{V} = yz \cosh(xyz)$$

and

$$\operatorname{rot} \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sinh(xyz) & z & x \end{vmatrix} = (-1, xy \cosh(xyz) - 1, -xz \cos(xyz)).$$

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8) If $\mathbf{V} = (\text{Arctan } z, \text{Arctan } x, \text{Arctan } y)$ then

$$\text{div } \mathbf{V} = 0$$

and

$$\text{rot } \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \text{Arctan } z & \text{Arctan } x & \text{Arctan } y \end{vmatrix} = \left(\frac{1}{1+y^2}, \frac{1}{1+z^2}, \frac{1}{1+x^2} \right).$$

Example 3.3 Find the divergence and the rotation of the vector field (the so-called Coulomb vector field),

$$\mathbf{V}(x, y, z) = \frac{1}{r^3}(x, y, z), \quad (x, y, z) \neq (0, 0, 0), \quad r = \sqrt{x^2 + y^2 + z^2}.$$

[Cf. **Example 4.3**]

A Divergence and rotation.

D Compute $\text{div } \mathbf{V} = \nabla \cdot \mathbf{V}$ and $\text{rot } \mathbf{V} = \nabla \times \mathbf{V}$.

I First notice that

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r} \quad \text{og} \quad \frac{\partial r}{\partial z} = \frac{z}{r}.$$

These are easy rules of calculations, by which

$$\begin{aligned} \text{div } \mathbf{V} &= \frac{\partial}{\partial x} \left(\frac{x}{r^3} \right) + \frac{\partial}{\partial y} \left(\frac{y}{r^3} \right) + \frac{\partial}{\partial z} \left(\frac{z}{r^3} \right) \\ &= \left(\frac{1}{r^3} - \frac{3x}{r^4} \frac{\partial r}{\partial x} \right) + \left(\frac{1}{r^3} - \frac{3y}{r^4} \frac{\partial r}{\partial y} \right) + \left(\frac{1}{r^3} - \frac{3z}{r^4} \frac{\partial r}{\partial z} \right) \\ &= \frac{3}{r^3} - \frac{3}{r^4} \cdot \frac{1}{r} (x^2 + y^2 + z^2) = \frac{3}{r^3} - \frac{3}{r^3} = 0, \end{aligned}$$

and

$$\begin{aligned}
 \operatorname{rot} \mathbf{V} &= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{r^3} & \frac{y}{r^3} & \frac{z}{r^3} \end{vmatrix} \\
 &= \left\{ z \frac{\partial}{\partial y} \left(\frac{1}{r^3} \right) - y \frac{\partial}{\partial z} \left(\frac{1}{r^3} \right) \right\} \mathbf{e}_x + \left\{ x \frac{\partial}{\partial z} \left(\frac{1}{r^3} \right) - z \frac{\partial}{\partial x} \left(\frac{1}{r^3} \right) \right\} \mathbf{e}_y \\
 &\quad + \left\{ y \frac{\partial}{\partial x} \left(\frac{1}{r^3} \right) - x \frac{\partial}{\partial y} \left(\frac{1}{r^3} \right) \right\} \mathbf{e}_z \\
 &= -\frac{3}{r^4} \left(z \frac{\partial r}{\partial y} - y \frac{\partial r}{\partial z}, x \frac{\partial r}{\partial z} - z \frac{\partial r}{\partial x}, y \frac{\partial r}{\partial x} - x \frac{\partial r}{\partial y} \right) \\
 &= -\frac{3}{r^4} \left(z \cdot \frac{y}{r} - y \cdot \frac{z}{r}, x \cdot \frac{z}{r} - z \cdot \frac{x}{r}, y \cdot \frac{x}{r} - x \cdot \frac{y}{r} \right) = (0, 0, 0).
 \end{aligned}$$

REMARK. A variant is

$$\begin{aligned}
 \operatorname{rot} \mathbf{V} &= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{r^3} & \frac{y}{r^3} & \frac{z}{r^3} \end{vmatrix} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} \left(\frac{1}{r^3} \right) & \frac{\partial}{\partial y} \left(\frac{1}{r^3} \right) & \frac{\partial}{\partial z} \left(\frac{1}{r^3} \right) \\ x & y & z \end{vmatrix} \\
 &= -\frac{3}{r^4} \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \\ x & y & z \end{vmatrix} = -\frac{3}{r^5} \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ x & y & x \\ x & y & z \end{vmatrix} = -\frac{3}{r^5} \mathbf{x} \times \mathbf{x} = \mathbf{0}. \quad \diamond
 \end{aligned}$$

Example 3.4 Choose the constants α and β , such that the vector field

$$\mathbf{V}(x, y, z) = (xyz)^\beta (x^\alpha, y^\alpha, z^\alpha), \quad (x, y, z) \in \mathbb{R}_+^3,$$

has zero rotation.

A Rotation free vector field.

D Compute $\operatorname{rot} \mathbf{V}$.

I We get by a calculation,

$$\begin{aligned}
 \mathbf{rot} \mathbf{V} &= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^{\alpha+\beta}y^\beta z^\beta & x^\beta y^{\alpha+\beta} & x^\beta y^\beta z^{\alpha+\beta} \end{vmatrix} \\
 &= \beta (x^\beta y^{\beta-1} z^{\alpha+\beta} - x^\beta y^{\alpha+\beta} z^{\beta-1}) \mathbf{e}_x + \beta (x^{\alpha+\beta} y^\beta z^{\beta-1} - x^{\beta-1} y^\beta z^{\alpha+\beta}) \mathbf{e}_y \\
 &\quad + \beta (x^{\beta-1} y^{\alpha+\beta} z^\beta - x^{\alpha+\beta} y^{\beta-1} z^\beta) \mathbf{e}_z \\
 &= \beta (xyz)^\beta (y^{-1} z^\alpha - y^\beta z^{-1}, x^\alpha z^{-1} - x^{-1} z^\alpha, x^{-1} y^\alpha - x^\alpha y^{-1}).
 \end{aligned}$$

If $\beta = 0$, then the factor outside the vector is 0, and the vector field becomes rotation free in \mathbb{R}_+^3 . This corresponds to the vector field

$$\mathbf{V}(x, y, z) = (x^\alpha, y^\alpha, z^\alpha), \quad \alpha \in \mathbb{R},$$

where the condition $(x, y, z) \in \mathbb{R}_+^3$ assures that the vector is always defined.

The second possibility is that the vector is

$$(y^{-1} z^\alpha - y^\alpha z^{-1}, x^\alpha z^{-1} - x^{-1} z^\alpha, x^{-1} y^\alpha - x^\alpha y^{-1}) = \mathbf{0}.$$

This gives the condition $\alpha = -1$, in which case the vector field becomes

$$\mathbf{V}(x, y, z) = (x^{\beta-1} y^\beta z^\beta, x^\beta y^{\beta-1} z^\beta, x^\beta y^\beta z^{\beta-1}) = (xyz)^\beta \left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right)$$

which is also free of rotation.

4 Gauß's theorem

Example 4.1 Find in each of the following cases the flux of the given vector field \mathbf{V} through the surface of the given set Ω in the space.

- 1) The vector field $\mathbf{V}(x, y, z) = (5xz, y^2 - 2yz, 2yz)$, defined in the domain Ω by $x^2 + y^2 \leq a^2$, $y \geq 0$, $0 \leq z \leq b$.
- 2) The vector field $\mathbf{V}(x, y, z) = (2x - \sqrt{1+z^2}, x^2y, -xz^2)$, defined in the cube $\Omega = [0, 1] \times [0, 1] \times [0, 1]$.
- 3) The vector field $\mathbf{V}(x, y, z) = (x^2+y^2, y^2+z^2, z^2+x^2)$ given in the domain Ω defined by $x^2+y^2+z^2 \leq a^2$ and $z \geq 0$.
- 4) The vector field $\mathbf{V}(x, y, z) = (2x + \sqrt[3]{y^2+z^2}, y - \cosh(xz), y^2 + 2z)$, defined in the domain $\Omega = \overline{K}((3, -1, 2); 3)$.
- 5) The vector field $\mathbf{V}(x, y, z) = (-x + \cos z, -xy, 3z + e^y)$, defined in the domain Ω given by $x \in [0, 3]$, $y \in [0, 2]$, $z \in [0, y^2]$.
- 6) The vector field ∇T , where $T(x, y, z) = x^2 + y^2 + z^2$ is defined in the domain Ω given by $x^2 + y^2 \leq 2$ and $z \in [0, 2]$.
- 7) The vector field $\mathbf{V}(x, y, z) = (x^3 + xy^2, 4yz^2 - 2x^2y, -z^3)$, defined in the ball given by $x^2 + y^2 + z^2 \leq a^2$.
- 8) The vector field $\mathbf{V}(x, y, z) = (2x, 3y, -z)$, defined in the ellipsoid Ω , given by $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \leq 1$.

A Flux out of a body in space.

D Apply Gauß's theorem of divergence.

I According to Gauß's theorem the flux is given by

$$\int_{\partial\Omega} \mathbf{V} \cdot \mathbf{n} dS = \int_{\Omega} \operatorname{div} \mathbf{V} d\Omega.$$

1) Since

$$\operatorname{div} \mathbf{V} = 5z + 2y - 2z + 2y = 3z + 4y,$$

the flux is

$$\int_{\Omega} \operatorname{div} \mathbf{V} d\Omega = \int_0^b 3z dz \cdot \frac{1}{2} \pi a^2 + 4 \int_0^b \int_0^\pi \left\{ \int_0^a \varrho \sin \varphi \cdot \varrho d\varrho \right\} d\varphi dz = \frac{3}{4} \pi a^2 b^2 + \frac{8}{3} a^3 b.$$

2) Since

$$\operatorname{div} \mathbf{V} = 2 + x^2 - 2xz,$$

the flux is

$$\int_{\Omega} \operatorname{div} \mathbf{V} d\Omega = 2 + \int_{\Omega} x^2 d\Omega - \int_{\Omega} 2xz d\Omega = 2 + \frac{1}{3} - \frac{1}{2} = \frac{11}{6}.$$

3) Here

$$\operatorname{div} \mathbf{V} = 2x + 2y + 2z.$$

It follows by the symmetry that

$$\int_{\Omega} 2x \, d\Omega = \int_{\Omega} 2y \, d\Omega = 0.$$

We obtain the flux by an application of Gauß's theorem, the argument of symmetry above and semi polar coordinate,

$$\begin{aligned} \int_{\Omega} \operatorname{div} \mathbf{V} \, d\Omega &= \int_{\Omega} 2x \, d\Omega + \int_{\Omega} 2y \, d\Omega + \int_{\Omega} 2z \, d\Omega = \int_{\Omega} 2z \, d\Omega \\ &= \int_0^{2\pi} \left\{ \int_0^a \left\{ \int_0^{\sqrt{a^2-\varrho^2}} 2z \, dz \right\} \varrho \, d\varrho \right\} d\varphi \\ &= 2\pi \int_0^a (a^2 - \varrho^2) \varrho \, d\varrho = 2\pi \left[\frac{a^2}{2} \varrho^2 - \frac{\varrho^4}{4} \right]_0^a = 2\pi \cdot \frac{a^4}{4} = \frac{\pi a^4}{2}. \end{aligned}$$

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4) Since

$$\operatorname{div} \mathbf{V} = 2 + 1 + 2 = 5,$$

the flux is

$$\int_{\Omega} \operatorname{div} \mathbf{V} d\Omega = 5 \operatorname{vol}(\overline{K}((3, -1, 2); 3)) = 5 \cdot \frac{4\pi}{3} \cdot 3^3 = 180\pi.$$

5) Since

$$\operatorname{div} \mathbf{V} = -1 - x + 3 = 2 - x,$$

the flux is given by

$$\begin{aligned} \int_{\Omega} \operatorname{div} \mathbf{V} d\Omega &= \int_{\Omega} (2 - x) d\Omega = \int_0^3 (2 - x) \left\{ \int_0^2 \left\{ \int_0^{y^2} dz \right\} dy \right\} dx \\ &= \left[2x - \frac{x^2}{2} \right]_0^3 \cdot \int_0^2 y^2 dy = \left(6 - \frac{9}{2} \right) \cdot \left[\frac{y^3}{3} \right]_0^2 = \frac{3}{2} \cdot \frac{8}{3} = 4. \end{aligned}$$

6) Since

$$\operatorname{div} \mathbf{V} = \Delta(x^2 + y^2 + z^2) = 2 + 2 + 2 = 6,$$

the flux is given by

$$\int_{\Omega} \operatorname{div} \mathbf{V} d\Omega = 6 \operatorname{vol}(\Omega) = 6 \cdot \pi \cdot (\sqrt{2})^2 \cdot 2 = 24\pi.$$

7) Here,

$$\operatorname{div} \mathbf{V} = 3x^2 + y^2 + 4z^2 - 2x^2 - 3z^2 = x^2 + y^2 + z^2.$$

The flux is easiest computed in spherical coordinates,

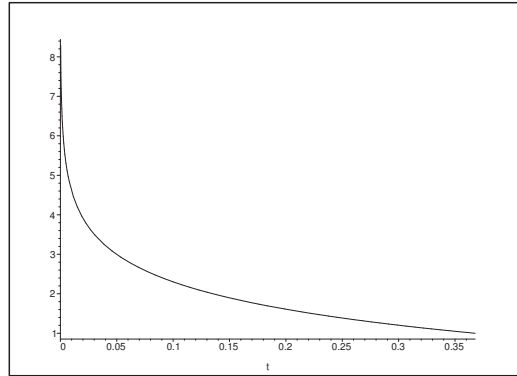
$$\int_{\Omega} \operatorname{div} \mathbf{V} d\Omega = \int_0^{2\pi} \left\{ \int_0^{\pi} \left\{ \int_0^a r^2 \cdot r \sin \theta dr \right\} d\theta \right\} d\varphi = 2\pi \left[\frac{r^5}{5} \right]_0^1 \cdot [-\cos \theta]_0^{\pi} = \frac{4}{5} \pi a^5.$$

8) From

$$\operatorname{div} \mathbf{V} = 2 + 3 - 1 = 4,$$

follows that the flux is

$$\int_{\Omega} \operatorname{div} \mathbf{V} d\Omega = 4 \operatorname{vol}(\Omega) = 4 \cdot \frac{4\pi}{3} abc = \frac{16}{3} \pi abc.$$

Figure 13: The meridian cut of i **Example 4.2.2**.

Example 4.2 Find in each of the following cases the flux of the given vector field \mathbf{V} through the surface of the described body of revolution Ω .

1) The vector field is $\mathbf{V}(x, y, z) = (y^2 + z^4, (x - a)^2 + z^4, x^2 + y^2)$, and the meridian cut of Ω is given by $\rho \leq a$ and $0 \leq z \leq \sqrt[4]{a^2 - \rho^2}$.

2) The vector field is

$$\mathbf{V}(x, y, z) = (x^2 - 2xy, 2y^2 + 6x^2z^2, 2z - 2xz - 2yz),$$

and the meridian cut of Ω is given by $0 \leq z \leq 1$ and $\rho \leq e^{-z}$.

3) The vector field is $\mathbf{V}(x, y, z) = (x^2 - xz, y^2 - yz, z^2)$, and the meridian cut of Ω is given by $\rho \leq \sqrt{\ln z}$ and $z \in [e, e^2]$.

4) The vector field is $\mathbf{V}(x, y, z) = (2x + 2y, 2y + z, z + 2x)$, and the meridian cut of Ω is given by

$$\rho \leq a, \quad \frac{\rho^2 - a^2}{a} \leq z \leq \sqrt{a^2 - \rho^2}.$$

A Flux from the surface of a body of revolution.

D Sketch if possible the meridian cut. Compute $\operatorname{div} \mathbf{V}$ and apply Gauß's theorem.

I 1) From $\operatorname{div} \mathbf{V} = 0$, follows trivially that the flux is

$$\int_{\Omega} \operatorname{div} \mathbf{V} d\Omega = 0,$$

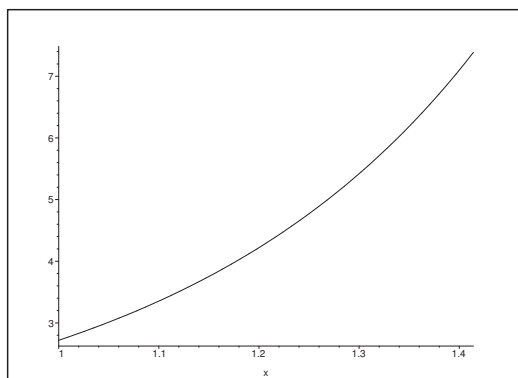
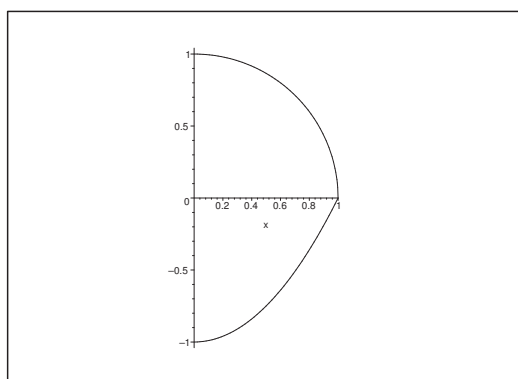
and we do not have to think about the body of revolution at all.

2) We conclude from

$$\operatorname{div} \mathbf{V} = 2x - 2y + 4y + 2 - 2x - 2y = 2,$$

that the flux is

$$\int_{\Omega} \operatorname{div} \mathbf{V} d\Omega = 2 \operatorname{vol}(\Omega) = 2 \int_0^1 \pi e^{-2z} dz = \pi (1 - e^{-2}).$$

Figure 14: The meridian cut of **Example 4.2.3**.Figure 15: The meridian cut of **Example 4.2.4**.

3) Here,

$$\operatorname{div} \mathbf{V} = 2x - z + 2y - z + 2z = 2x + 2y.$$

If we put

$$B(z) = \{(x, y) \mid x^2 + y^2 \leq \ln z\}, \quad z \in [e, e^2],$$

then the flux is

$$\int_{\Omega} \operatorname{div} \mathbf{V} \, d\Omega = \int_{\Omega} (2x + 2y) \, d\Omega = \int_e^{e^2} \left\{ \int_{B(z)} (2x + 2y) \, dx \, dy \right\} dz = 0,$$

because it follows from the symmetry that

$$\int_{B(z)} x \, dx \, dy = \int_{B(z)} y \, dx \, dy = 0.$$

4) It follows from the equations of the meridian cut that when $z > 0$ we have the quarter of a circle, and when $z < 0$ we get an arc of a parabola. It is natural to split the cut of Ω_0 correspondingly in Ω_1 (for $z > 0$) and Ω_2 (for $z < 0$).

Since

$$\operatorname{div} \mathbf{V} = 2 + 2 + 1 = 5,$$

we get by Gauß's theorem that the flux is

$$\begin{aligned} \text{flux} &= \int_{\partial\Omega} \mathbf{V} \cdot \mathbf{n} \, dS = \int_{\Omega} \operatorname{div} \mathbf{V} \, d\Omega = 5 \operatorname{vol}(\Omega) = 5 \operatorname{vol}(\Omega_1) + 5 \operatorname{vol}(\Omega_2) \\ &= 5 \cdot \frac{1}{2} \cdot \frac{4\pi}{3} a^3 + 5 \int_{-a}^0 \pi \varrho(z)^2 \, dz = \frac{10\pi}{3} a^3 + 5\pi \int_{-a}^0 (az + a^2) \, dz \\ &= \frac{10\pi}{3} a^3 + 5\pi \left[\frac{az^2}{2} + a^2z \right]_{-a}^0 = \frac{10\pi}{3} a^3 + 5\pi \left(-\frac{a^3}{2} + a^3 \right) \\ &= 5\pi a^3 \left(\frac{2}{3} + \frac{1}{2} \right) = 5\pi a^3 \cdot \frac{7}{6} = \frac{35}{6} \pi a^3. \end{aligned}$$

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Example 4.3 Let Ω denote the cylinder given by $z \in [-h, h]$, $\varrho \in [0, a]$, $\varphi \in [0, 2\pi]$. Find the flux through the surface $\partial\Omega$ of the Coulomb vector field

$$\mathbf{V}(x, y, z) = \frac{1}{r^3} (x, y, z), \quad (x, y, z) \neq (0, 0, 0), \quad r = \sqrt{x^2 + y^2 + z^2}.$$

[Cf. **Example 2.1.5**, **Example 2.1.6** and **Example 3.3**].

A Flux through the surface of a body.

D Think of how to treat the singularity at $(0, 0, 0)$ before we can apply Gauß's theorem. Find the flux.

I When $(x, y, z) \neq (0, 0, 0)$, we get [cf. **Example 3.3**]

$$\frac{\partial V_1}{\partial x} = \frac{1}{r^3} - \frac{3}{r^5} x^2, \quad \frac{\partial V_2}{\partial y} = \frac{1}{r^3} - \frac{3}{r^5} y^2, \quad \frac{\partial V_3}{\partial z} = \frac{1}{r^3} - \frac{3}{r^5} z^2,$$

hence

$$\operatorname{div} \mathbf{V} = \frac{3}{r^3} - \frac{3}{r^5} (x^2 + y^2 + z^2) = \frac{3}{r^3} - \frac{3}{r^5} r^2 = 0.$$

One could therefore be misled to “conclude” that the flux is 0, “because $(0, 0, 0)$ is a null set”; but this is not true.

Let $R \in]0, \min\{a, h\}[$. An application of Gauß's theorem shows that the flux through the surface of $\Omega \setminus K(\mathbf{0}; R)$ is

$$\int_{\Omega \setminus K(\mathbf{0}; R)} \operatorname{div} \mathbf{V} \, d\Omega = 0,$$

because $(0, 0, 0) \notin \Omega \setminus K(\mathbf{0}; R)$. Hence, the flux is

$$\begin{aligned} \int_{\partial\Omega} \mathbf{V} \cdot \mathbf{n} \, dS &= \left\{ \int_{\partial\Omega} \mathbf{V} \cdot \mathbf{n} \, dS - \int_{\partial K(\mathbf{0}; R)} \mathbf{V} \cdot \mathbf{n} \, dS \right\} + \int_{\partial K(\mathbf{0}; R)} \mathbf{V} \cdot \mathbf{n} \, dS \\ &= \int_{\Omega \setminus K(\mathbf{0}; R)} \operatorname{div} \mathbf{V} \, d\Omega + \int_{\partial K(\mathbf{0}; R)} \mathbf{V} \cdot \mathbf{n} \, dS = \int_{\partial K(\mathbf{0}; R)} \mathbf{V} \cdot \mathbf{n} \, dS. \end{aligned}$$

On the boundary $\partial K(\mathbf{0}; R)$ the outer unit normal vector is given in rectangular coordinates by $\mathbf{n} = \frac{1}{R} (x, y, z)$, thus

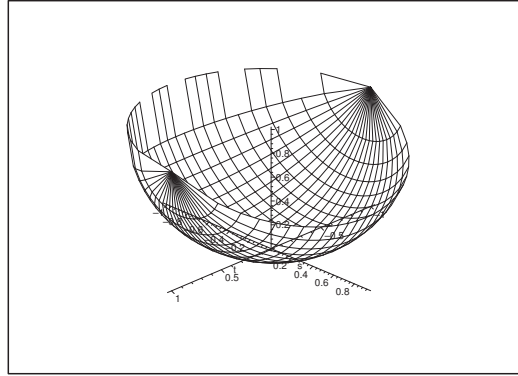
$$\mathbf{V} \cdot \mathbf{n} = \frac{1}{R^3} (x, y, z) \cdot \frac{1}{R} (x, y, z) = \frac{1}{R^2}.$$

The area element is given in polar coordinates by

$$dS = R^2 \sin \theta \, d\theta \, d\varphi.$$

Then the flux through $\partial\Omega$ is given by

$$\int_{\partial\Omega} \mathbf{V} \cdot \mathbf{n} \, dS = \int_{\partial K(\mathbf{0}; R)} \mathbf{V} \cdot \mathbf{n} \, dS = \int_0^{2\pi} \left\{ \int_0^\pi \frac{1}{R^2} \cdot R^2 \sin \theta \, d\theta \right\} d\varphi = 2\pi [-\cos \theta]_0^\pi = 4\pi,$$

Figure 16: The surface of **Example 4.4** for $a = 1$.

Example 4.4 We shall find the flux Φ of the vector field

$$\mathbf{V}(x, y, z) = (e^y + \cosh z, e^x + \sinh z, x^2 z^2), \quad (x, y, z) \in \mathbb{R}^3,$$

through the oriented half sphere \mathcal{F} given by

$$x^2 + y^2 + z^2 - 2az = 0, \quad z \leq a, \quad \mathbf{n} \cdot \mathbf{e}_z \geq 0.$$

It turns up that the integration over \mathcal{F} is rather difficult, while on the other hand the expression of $\operatorname{div} \mathbf{V}$ is fairly simple. One shall therefore try to arrange the calculations such that it becomes possible to apply Gauß's theorem.

- 1) Construct a closed surface by adding an oriented disc \mathcal{F}_1 to \mathcal{F} . Sketch the meridian half plane.
- 2) Find the flux Φ_1 of the vector field \mathbf{V} through \mathcal{F}_1 .
- 3) Apply Gauß's theorem on the body Ω of the boundary $\partial\Omega = \mathcal{F} \cup \mathcal{F}_1$, and then find Φ .

A Computation of the flux of a vector field through a surface where a direct calculation becomes very difficult.

D Apply the guidelines, i.e. add a surface \mathcal{F}_1 , such that $\mathcal{F} \cup \mathcal{F}_1$ surrounds a body, on which Gauß's theorem can be applied. Hence, something is added and then subtracted again, and then one uses Gauß's theorem.

I 1) When we add a^2 to both sides of the equation of the half sphere, we obtain

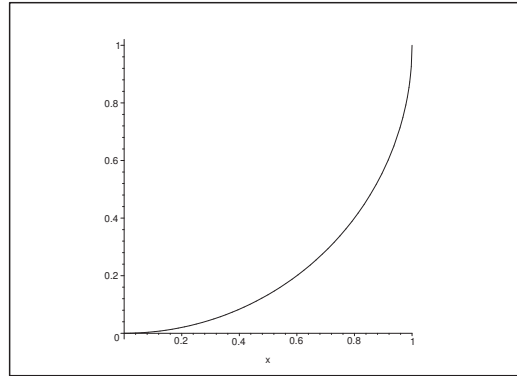
$$a^2 = x^2 + y^2 + z^2 - 2az + a^2 = \varrho^2 + (z - a)^2.$$

It follows from the condition $\mathbf{n} \cdot \mathbf{e}_z \geq 0$ that the curve in the meridian half plane of \mathcal{F} is the quarter of a circle of centrum $(0, a)$ and radius a ,

$$\varrho^2 + (z - a)^2 = a^2, \quad z \leq a, \quad \varrho \geq 0.$$

Note that the normal vector has an upwards pointing component.

The disc ("the lid"), which shall be added is of course the disc in the plane $z = a$ of centrum $(0, 0, a)$ and radius a .

Figure 17: The meridian curve of **Example 4.4** for $a = 1$.

2) The flux of \mathbf{V} through \mathcal{F}_1 of normal \mathbf{e}_z is

$$\int_{\mathcal{F}_1} \mathbf{V} \cdot \mathbf{n} dS = \int_{\mathcal{F}_1} x^2 a^2 dS = a^2 \int_0^{2\pi} \cos^2 \varphi \left\{ \int_0^a \varrho^2 \cdot \varrho d\varrho \right\} d\varphi = a^2 \pi \cdot \left[\frac{\varrho^4}{4} \right]_0^a = \frac{\pi}{4} a^6.$$

3) Let Ω be the domain which is surrounded by $\mathcal{F}_1 \cup (-\mathcal{F})$, where $-\mathcal{F}$ indicates that we have reversed the orientation, such that the normal is pointing away from Ω) on both \mathcal{F}_1 and $-\mathcal{F}$.

Now

$$\operatorname{div} \mathbf{V} = 0 + 0 + 2x^2 z = 2x^2 z = 2x^2(z - a) + 2ax^2,$$

so it follows by Gauß's theorem that

$$-\int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n} dS + \int_{\mathcal{F}_1} \mathbf{V} \cdot \mathbf{n} dS = -\int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n} dS + \frac{\pi}{4} a^6 = \int_{\Omega} \operatorname{div} \mathbf{V} d\Omega,$$

hence by a rearrangement,

$$\begin{aligned} \Phi &= \int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n} dS = \frac{\pi}{4} a^6 - \int_{\Omega} \operatorname{div} \mathbf{V} d\Omega = \frac{\pi}{4} a^6 - \int_{\Omega} 2ax^2 d\Omega - \int_{\Omega} 2x^2(z - a) d\Omega \\ &= \frac{\pi}{4} a^6 - a \int_{\Omega} d\Omega + \int_{\Omega} (x^2 + y^2)(a - z) d\Omega, \end{aligned}$$

where we have used the symmetry in x and y in the domain of integration in the latter equality.

By the transformation $z \rightsquigarrow a - z$ the half ball Ω is mapped into the half ball

$$\Omega_1 = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq a^2, z \geq 0\},$$

so

$$\Phi = \frac{\pi}{4} a^6 - a \int_{\Omega_1} (x^2 + y^2) d\Omega + \int_{\Omega_1} (x^2 + y^2)z d\Omega.$$

When we use the slicing method, we see that Ω_1 at height $z \in [0, a]$ is cut into the circle

$$B(z) = \{(x, y, z) \mid x^2 + y^2 \leq a^2 - z^2\} = \{(x, y, z) \mid \varrho \leq \sqrt{a^2 - z^2}\}, \quad z \in [0, a] \text{ fixed},$$

thus

$$\begin{aligned}
 a \int_{\Omega_1} (x^2 + y^2) d\Omega &= a \int_0^a \left\{ \int_{B(z)} (x^2 + y^2) dS \right\} dz \\
 &= a \int_0^a \left\{ \int_0^{2\pi} \left[\int_0^{\sqrt{a^2-z^2}} \varrho^2 \cdot \varrho d\varrho \right] d\varphi \right\} dz = 2\pi a \int_0^a \left[\frac{\varrho^4}{4} \right]_0^{\sqrt{a^2-z^2}} dz \\
 &= \frac{\pi}{2} a \int_0^a (a^2 - z^2)^2 dz = \frac{\pi}{2} a \int_0^a (z^4 - 2a^2 z^2 + a^4) dz \\
 &= \frac{\pi}{2} a \left[\frac{z^5}{5} - \frac{2a^2}{3} z^3 + a^4 z \right]_0^a = \frac{\pi}{2} a \left\{ \frac{a^5}{5} - \frac{2}{3} a^5 + a^5 \right\} \\
 &= \frac{\pi}{2} a^6 \cdot \left(\frac{1}{5} - \frac{2}{3} + 1 \right) = \frac{4\pi}{15} a^6,
 \end{aligned}$$

and by some reuse of previous results,

$$\begin{aligned}
 \int_{\Omega_1} (x^2 + y^2) z d\Omega &= \int_0^a z \left\{ \int_{B(z)} (x^2 + y^2) dS \right\} dz \\
 &= \frac{\pi}{2} \int_0^a (z^2 - a^2)^2 \cdot z dz = \frac{\pi}{4} \left[\frac{1}{3} (z^2 - a^2)^3 \right]_0^a = \frac{\pi}{12} a^6.
 \end{aligned}$$

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Finally, we get by insertion that

$$\begin{aligned} \Phi &= \frac{\pi}{4} a^6 - a \int_{\Omega_1} (x^2 + y^2) d\Omega + \int_{\Omega_1} (x^2 + y^2) z d\Omega \\ &= \frac{\pi}{4} a^6 - \frac{4\pi}{15} a^6 + \frac{\pi}{12} a^6 = \frac{\pi a^6}{60} (15 - 16 + 5) = \frac{\pi a^6}{15}. \end{aligned}$$

Example 4.5 Let a set $\Omega \subset \mathbb{R}^3$ and a vector field $\mathbf{V} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be given in the following way,

$$\Omega = \left\{ (x, y, z) \mid \frac{x^2 + y^2 - a^2}{a} \leq z \leq \sqrt{a^2 - x^2 - y^2} \right\},$$

$$\mathbf{V}(x, y, z) = (2x + 2y, 2y + z, z + 2x).$$

The boundary $\partial\Omega$ is oriented such that the normal vector is always pointing away from the body. By \mathcal{F}_1 and \mathcal{F}_2 we denote the subsets of $\partial\Omega$, for which $z \geq 0$, and $z \leq 0$, respectively. Find the fluxes of \mathbf{V} through \mathcal{F}_1 and \mathcal{F}_2 , respectively.

A Flux through surfaces.

D Apply both rectangular and polar coordinates. Check Gauß's theorem. This cannot be applied directly. It can, however, come into play by a small extra argument.

Fiannly, compute the fluxes.

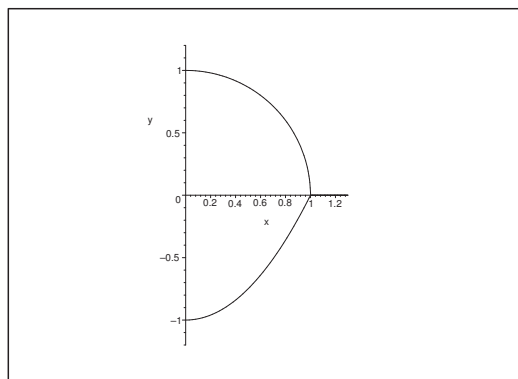


Figure 18: The cut of the meridian half plane for $a = 1$.

I By using semi polar coordinates we obtain that

$$az \geq \varrho^2 - a^2 \quad \text{og} \quad z^2 + \varrho^2 \leq a^2,$$

and the meridian half plane becomes like shown on the figure.

As

$$\begin{aligned} \text{vol}(\Omega) &= \text{vol}(\Omega_1) + \text{vol}(\Omega_2) = \frac{1}{2} \cdot \frac{4\pi}{3} a^3 + \int_{-a}^0 \pi \varrho(z)^2 dz = \frac{2\pi}{3} a^3 + \pi \int_{-a}^0 a(a+z) dz \\ &= \frac{2\pi}{3} a^3 + \frac{\pi}{2} a \int_0^a 2t dt = \frac{2\pi}{3} a^3 + \frac{\pi}{2} a^3 = \frac{7\pi}{6} a^3, \end{aligned}$$

and $\operatorname{div} \mathbf{V} = 2 + 2 + 1 = 5$, it follows from Gauß's theorem that

$$\operatorname{flux}(\mathcal{F}) = \operatorname{flux}(\mathcal{F}_1) + \operatorname{flux}(\mathcal{F}_2) = \int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n} \, dS = \int_{\Omega} \operatorname{div} \mathbf{V} \, d\Omega = 5 \operatorname{vol}(\Omega) = \frac{35\pi}{6} a^3.$$

The parametric description of \mathcal{F}_1 is chosen as

$$\mathbf{r}(u, v) = \left(u, v, \sqrt{a^2 - u^2 - v^2} \right), \quad u^2 + v^2 \leq a^2,$$

thus

$$\frac{\partial \mathbf{r}}{\partial u} = \left(1, 0, -\frac{u}{\sqrt{a^2 - u^2 - v^2}} \right) \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial v} = \left(0, 1, -\frac{v}{\sqrt{a^2 - u^2 - v^2}} \right),$$

from which we get the normal vector

$$\mathbf{N}(u, v) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 1 & 0 & -\frac{u}{\sqrt{a^2 - u^2 - v^2}} \\ 0 & 1 & -\frac{v}{\sqrt{a^2 - u^2 - v^2}} \end{vmatrix} = \frac{1}{\sqrt{a^2 - u^2 - v^2}} \left(u, v, \sqrt{a^2 - u^2 - v^2} \right),$$

which is clearly pointing away from the body, because the Z -coordinate is $+1$.

If we put $B = \{(u, v) \mid u^2 + v^2 < a^2\}$, it follows from $(x, y, z) = (u, v, \sqrt{a^2 - u^2 - v^2})$ that

$$\begin{aligned} \operatorname{flux}(\mathcal{F}_1) &= \int_{\mathcal{F}_1} \mathbf{V} \cdot \mathbf{n} \, dS = \int_B \mathbf{V}(u, v) \cdot \mathbf{N}(u, v) \, du \, dv \\ &= \int_B (2u + 2v, 2v + \sqrt{a^2 - u^2 - v^2}, \sqrt{a^2 - u^2 - v^2} + 2u) \\ &\quad \cdot \frac{1}{\sqrt{a^2 - u^2 - v^2}} (u, v, \sqrt{a^2 - u^2 - v^2}) \, du \, dv \\ &= \int_B \frac{1}{\sqrt{a^2 - u^2 - v^2}} \{2u^2 + 2uv + 2v^2 + v\sqrt{a^2 - u^2 - v^2} \\ &\quad + (a^2 - u^2 - v^2) + 2u\sqrt{a^2 - u^2 - v^2}\} \, du \, dv \\ &= \int_B \frac{a^2 + u^2 + v^2}{\sqrt{a^2 - u^2 - v^2}} \, du \, dv + 0 = \int_0^{2\pi} \left\{ \int_0^a \frac{a^2 + \varrho^2}{\sqrt{a^2 - \varrho^2}} \cdot \varrho \, d\varrho \right\} d\varphi = \pi \int_0^{a^2} \frac{a^2 + t}{\sqrt{a^2 - t}} \, dt \\ &= \pi \int_0^{a^2} \left\{ \frac{2a^2}{\sqrt{a^2 - t}} - \sqrt{a^2 - t} \right\} \, dt = \pi \left[-4a^2 \sqrt{a^2 - t} + \frac{2}{3} (\sqrt{a^2 - t})^3 \right]_0^{a^2} \\ &= \pi \left\{ 4a^2 \sqrt{a^2} - \frac{2}{3} a^3 \right\} = \frac{10\pi}{3} a^3. \end{aligned}$$

Hence

$$\operatorname{flux}(\mathcal{F}_2) = \operatorname{flux}(\mathcal{F}) - \operatorname{flux}(\mathcal{F}_1) = \frac{35\pi}{6} a^3 - \frac{10\pi}{3} a^3 = \frac{5}{2} \pi a^3,$$

and thus

$$\text{flux}(\mathcal{F}_1) = \frac{10\pi}{3} a^3 \quad \text{and} \quad \text{flux}(\mathcal{F}_2) = \frac{5\pi}{2} a^3.$$

ALTERNATIVELY, \mathcal{F}_2 is given by the parametric description

$$\mathbf{r} = (x, y, z) = \left(u, v, \frac{1}{a}(u^2 + v^2 - a^2) \right), \quad (u, v) \in B,$$

thus

$$\frac{\partial \mathbf{r}}{\partial u} = \left(1, 0, \frac{2u}{a} \right) \quad \text{og} \quad \frac{\partial \mathbf{r}}{\partial v} = \left(0, 1, \frac{2v}{a} \right)$$

and hence

$$\mathbf{N}_1(u, v) = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 1 & 0 & \frac{2u}{a} \\ 0 & 1 & \frac{2v}{a} \end{vmatrix} = \left(-\frac{2u}{a}, -\frac{2v}{a}, 1 \right).$$

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This normal vector is pointing inwards, so we are forced to choose

$$\mathbf{N}(u, v) = -\mathbf{N}_1(u, v) = \left(\frac{2u}{a}, \frac{2v}{a}, -1 \right).$$

Then

$$\begin{aligned} \text{flux}(\mathcal{F}_2) &= \int_{\mathcal{F}_2} \mathbf{V} \cdot \mathbf{n} \, dS = \int_B \mathbf{V}(u, v) \cdot \mathbf{N}(u, v) \, du \, dv \\ &= \int_B \left(2u+2v, 2v+\frac{1}{a}(u^2+v^2-a^2), \frac{1}{a}(u^2+v^2-a^2) \right) \cdot \left(\frac{2u}{a}, \frac{2v}{a}, -1 \right) \, du \, dv \\ &= \int_B \left\{ \frac{4u^2}{a} + \frac{4uv}{a} + \frac{4v^2}{a} + \frac{2v}{a}(u^2+v^2-a^2) - \frac{1}{a}(u^2+v^2-a^2) \right\} \, du \, dv \\ &= \frac{1}{a} \int_B \{4u^2 + 4v^2 - u^2 - v^2 + a^2\} \, du \, dv + 0 \\ &= \frac{a^2}{2} \text{areal}(B) + \frac{3}{a} \int_B (u^2 + v^2) \, du \, dv = a \cdot \pi a^2 + \frac{3}{a} \cdot 2\pi \int_0^a \varrho^2 \cdot \varrho \, d\varrho \\ &= \pi a^3 + \frac{6\pi}{a} \cdot \frac{a^4}{4} = \frac{5\pi}{2} a^3, \end{aligned}$$

in accordance with the previous found result.

Example 4.6 Let K be the ball $(\bar{\mathbf{x}}_0; a)$, and let \mathbf{V} be a C^1 vector field on A , where $A \supset K$. Prove the following claims by using partial integration, Gauß's divergence theorem and the formula

$$\mathbf{x} = \frac{1}{2} \nabla (\mathbf{x} \cdot \mathbf{x}).$$

1) If the divergence of \mathbf{V} is a constant p , then

$$\int_K (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{V}(\mathbf{x}) \, d\Omega = \frac{4}{15} a^5 p.$$

2) If the rotation of \mathbf{V} is a constant vector \mathbf{P} , then

$$\int_K (\mathbf{x} - \mathbf{x}_0) \times \mathbf{V}(\mathbf{x}) \, d\Omega = \frac{4}{15} a^5 \mathbf{P}.$$

A Generalized partial integration.

D Follow the guidelines.

I 1) It follows from

$$\mathbf{x} - \mathbf{x}_0 = \frac{1}{2} \nabla ((\mathbf{x} - \mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)) = \frac{1}{2} \nabla (\|\mathbf{x} - \mathbf{x}_0\|^2),$$

and $f(\mathbf{x}) = \|\mathbf{x} - \mathbf{x}_0\|^2$ that

$$\begin{aligned} \int_K (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{V}(\mathbf{x}) \, d\Omega &= \frac{1}{2} \int_K \nabla (\|\mathbf{x} - \mathbf{x}_0\|^2) \cdot \mathbf{V}(\mathbf{x}) \, d\Omega \\ &= \frac{1}{2} \int_{\partial K} \mathbf{n} \cdot \mathbf{V}(\mathbf{x}) \|\mathbf{x} - \mathbf{x}_0\|^2 \, dS - \frac{1}{2} \int_{\Omega} \|\mathbf{x} - \mathbf{x}_0\|^2 \nabla \cdot \mathbf{V} \, d\Omega \\ &= \frac{1}{2} a^2 \int_{\partial K} \mathbf{n} \cdot \mathbf{V}(\mathbf{x}) \, dS - \frac{1}{2} p \int_{\Omega} \|\mathbf{x} - \mathbf{x}_0\|^2 \, d\Omega \\ &= \frac{1}{2} a^2 \int_{\Omega} \nabla \cdot \mathbf{V}(\mathbf{x}) \, d\Omega - \frac{1}{2} p \int_0^a \left\{ \int_0^{2\pi} \left(\int_0^\pi r^2 \cdot r^2 \sin \theta \, d\theta \right) d\varphi \right\} dr \\ &= \frac{1}{2} p a^2 \cdot \text{vol}(\Omega) - \frac{1}{2} p \int_0^a r^r \, dr \cdot 2\pi \cdot \int_0^\pi \sin \theta \, d\theta \\ &= \frac{1}{2} p a^2 \cdot \frac{4\pi}{3} a^3 - \frac{1}{2} p \cdot \frac{a^5}{5} \cdot 2\pi \cdot 2 = \frac{p a^5 \pi}{15} \cdot \{10 - 6\} = \frac{4}{15} a^5 \pi p. \end{aligned}$$

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2) We can now replace \cdot by \times , hence

$$\begin{aligned} \int_K (\mathbf{x} - \mathbf{x}_0) \times \mathbf{V}(\mathbf{x}) \, d\Omega &= \frac{1}{2} \int_K \nabla (\|\mathbf{x} - \mathbf{x}_0\|^2) \times \mathbf{V}(\mathbf{x}) \, d\Omega \\ &= \frac{1}{2} \int_{\partial K} \mathbf{n} \times \mathbf{V}(\mathbf{x}) \|\mathbf{x} - \mathbf{x}_0\|^2 \, dS - \frac{1}{2} \int_{\Omega} \|\mathbf{x} - \mathbf{x}_0\|^2 \nabla \times \mathbf{V} \, d\Omega \\ &= \frac{1}{2} a^2 \int_{\Omega} \nabla \times \mathbf{V}(\mathbf{x}) \, d\Omega - \frac{1}{2} \mathbf{P} \int_0^a \left\{ \int_0^{2\pi} \left(\int_0^{\pi} r^2 \cdot r^2 \sin \theta \, d\theta \right) d\varphi \right\} dr \\ &= \frac{1}{2} a^2 \mathbf{P} \cdot \text{vol}(\Omega) - \frac{1}{2} \mathbf{P} \int_0^a r^4 \, dr \cdot 2\pi \cdot \int_0^{\pi} \sin \theta \, d\theta \\ &= \left\{ \frac{1}{2} a^2 \cdot \frac{4\pi}{3} a^3 - \frac{1}{2} a^5 \cdot 2\pi \cdot 2 \right\} \mathbf{P} = \frac{4}{15} a^5 \pi \mathbf{P}. \end{aligned}$$

Example 4.7 Let a be a positive constant. We let T denote the subset of

$$T_1 = \{(x, y, z) \in \mathbb{R}^3 \mid z \geq 0, x^2 + y^2 + z^2 \leq 9a^2\},$$

which also lies outside the set

$$T_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + (z - a)^2 < a^2\},$$

thus $T = T_1 \setminus T_2$.

1) Explain why T is given in spherical coordinates by

$$\theta \in \left[0, \frac{\pi}{2}\right], \quad \varphi \in [0, 2\pi], \quad r \in [2a \cos \theta, 3a].$$

2) Find the mass of T when the density of mass on T is $\mu(x, y, z) = \frac{z}{a^4}$.

3) Find the flux of the vector field

$$\mathbf{V}(x, y, z) = (xz + 4xy, yz - 2y^2, x^2y^2), \quad (x, y, z) \in \mathbb{R}^3,$$

through ∂T .

4) Find the volume of the subset T^* of T , which is given by the inequalities

$$x \geq 0, \quad y \geq 0, z \geq \sqrt{x^2 + y^2}.$$

A Spherical coordinates, mass, flux, volume.

D Sketch the meridian half plane; compute a space integral; apply Gauß's theorem; once again, consider the meridian half plane.

I 1) When we consider the meridian half plane, it follows immediately that

$$\theta \in \left[0, \frac{\pi}{2}\right] \quad \text{and} \quad \varphi \in [0, 2\pi].$$

It only remains to prove that the meridian cut of ∂T_2 has the equation

$$r = 2a \cos \theta.$$

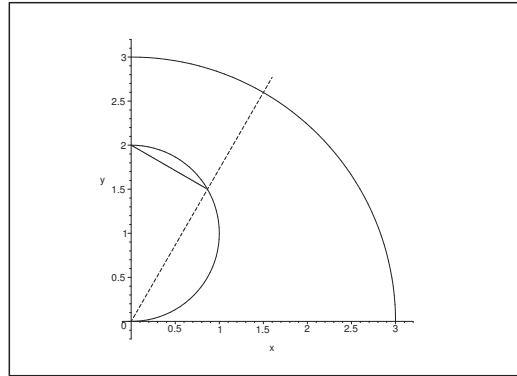


Figure 19: The meridian half plane for T , when $a = 1$. The angle between the Z -axis and the dotted radius is θ . The two dotted lines are perpendicular to each other.

Draw a radius and the perpendicular line on this as shown by the dotted lines on the figure. Together with the line segment $[0, 2a]$ on the Y -axis these form a rectangular triangle. The angle between the Z -axis and the dotted radius is θ , and the hypotenuse (the line segment on the Z -axis) is $2a$. Hence, the closest of the smaller sides (i.e. placed up to ∂T_2) must have the length $2a \cos \theta$. This proves that the equation of ∂T_2 is

$$r = 2a \cos \theta.$$

It then follows that $r \in [2a \cos \theta, 3a]$ in T .

2) We have in spherical coordinates

$$\mu(x, y, z) = \frac{z}{a^4} = \frac{r}{a^4} \cos \theta,$$

hence the mass

$$\begin{aligned} M &= \int_T \mu \, d\Omega = \int_0^{\frac{\pi}{2}} \left\{ \int_0^{2\pi} \left\{ \int_{2a \cos \theta}^{3a} \frac{1}{a^4} r \cos \theta \cdot r^2 \sin \theta \, dr \right\} d\varphi \right\} d\theta \\ &= \frac{2\pi}{a^4} \int_0^{\frac{\pi}{2}} \cos \theta \cdot \sin \theta \left[\frac{r^4}{4} \right]_{2a \cos \theta}^{3a} d\theta = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} (81 - 16 \cos^4 \theta) \cos \theta \sin \theta \, d\theta \\ &= \frac{\pi}{2} \left[-\frac{81}{2} \cos^2 \theta + \frac{16}{6} \cos^6 \theta \right]_0^{\frac{\pi}{2}} = \frac{\pi}{2} \left(\frac{81}{2} - \frac{16}{6} \right) = \frac{\pi}{12} (243 - 16) = \frac{117\pi}{12}. \end{aligned}$$

3) From

$$\operatorname{div} \mathbf{V} = z + 4y + z - 4y + 0 = 2z,$$

follows by Gauß's theorem and 2) that the flux is

$$\int_{\partial T} \mathbf{V} \cdot \mathbf{n} \, dS = \int_T \operatorname{div} \mathbf{V} \, d\Omega = \int_T 2z \, d\Omega = 2a^4 \int_T \mu \, d\Omega = \frac{227\pi}{6} a^4.$$

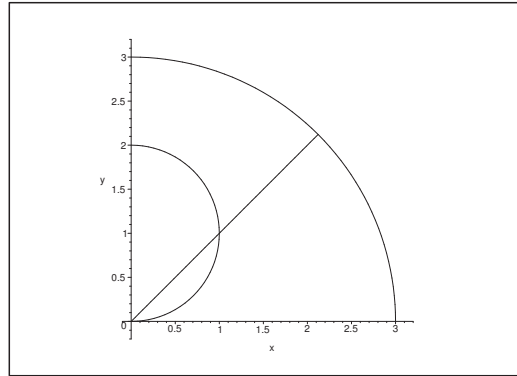


Figure 20: The meridian cut of T^* is the domain between the two circular arcs lying above the line $z = \varrho$.

4) By analyzing the meridian half plane once more we see that T^* is given by

$$\theta \in \left[0, \frac{\pi}{4}\right], \quad \varphi \in \left[+, \frac{\pi}{2}\right], \quad r \in [2a \cos \theta, 3a],$$

hence the volume is

$$\begin{aligned} \text{vol}(T^*) &= \int_0^{\frac{\pi}{4}} \left\{ \int_0^{\frac{\pi}{2}} \left\{ \int_{2a \cos \theta}^{3a} r^2 \sin \theta \, dr \right\} d\varphi \right\} d\theta = \frac{\pi}{2} \int_0^{\frac{\pi}{4}} \sin \theta \cdot \left[\frac{1}{3} r^3 \right]_{2a \cos \theta}^{3a} d\theta \\ &= \frac{\pi}{6} a^3 \int_0^{\frac{\pi}{4}} (27 - 8 \cos^3 \theta) \sin \theta \, d\theta = \frac{\pi}{6} a^3 [-27 \cos \theta + 2 \cos^4 \theta]_0^{\frac{\pi}{4}} \\ &= \frac{\pi}{6} a^3 \left(-\frac{27}{\sqrt{2}} + \frac{2}{4} + 27 - 2 \right) = \frac{\pi}{12} (51 - 27\sqrt{2}) a^3. \end{aligned}$$

Example 4.8 Let a be a positive constant and consider the function

$$f(x, y, z) = a^2 x^2 + a^3 y + z^4, \quad (x, y, z) \in \mathbb{R}^3.$$

1) Find the gradient $\mathbf{V} = \nabla f$ and the tangential line integral

$$\int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} \, ds,$$

where \mathcal{K} is the line segment from $(0, 0, a)$ to $(2a, 3a, 0)$.

2) Find the flux of \mathbf{V} through the surface of the half sphere given by

$$x^2 + y^2 + z^2 \leq a^2 \quad \text{and} \quad z \geq 0.$$

A Gradient; tangential line integral; flux.

D Apply Gauß's theorem in 2).

I 1) The gradient is

$$\mathbf{V} = \nabla f = (2a^2x, a^3, 4z^3).$$

Now, \mathbf{V} is a gradient field, $\mathbf{V} = \nabla f$, so

$$\int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} \, ds = f(2a, 3a, 0) - f(0, 0, a) = (a^2 \cdot 4a^2 + a^3 \cdot 3a) - a^4 = 6a^4.$$

2) We get by Gauß's theorem,

$$\begin{aligned} \text{flux}(\partial L) &= \int_{\partial L} \mathbf{V} \cdot \mathbf{n} \, dS = \int_L \text{div } \mathbf{V} \, d\Omega = \int_L (2a^2 + 12z^2) \, d\Omega \\ &= 2a^2 \cdot \frac{1}{2} \cdot \frac{4\pi}{3} a^3 + 12 \int_L z^2 \, d\Omega = \frac{4\pi}{3} a^5 + 12 \int_0^a z^2 \cdot \pi(a^2 - z^2) \, dz \\ &= \frac{4\pi}{3} a^5 + 12\pi \left[a^2 \cdot \frac{1}{3} z^3 - \frac{1}{5} z^5 \right]_0^a = \frac{4\pi}{3} a^5 + \frac{24}{15} \pi a^5 = \frac{44\pi}{15} a^5. \end{aligned}$$

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Example 4.9 Given the tetrahedron

$$T = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x, 0 \leq y, 4 - x - 2y \leq z \leq 8 - 2x - 4y\}.$$

and the vector field

$$\mathbf{V}(x, y, z) = \left(z \cos x + 3yz, x^2 y + x \sinh z, \frac{1}{2} z^2 \sin x + 3x^2 - 5y^2 \right), \quad (x, y, z) \in \mathbb{R}^3.$$

Find the flux of \mathbf{V} through ∂T .

A Flux of a vector field through a closed surface.

D Apply Gauß's theorem.

I It follows from

$$\operatorname{div} \mathbf{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} = -z \sin x + x^3 + \frac{1}{2} \cdot 2z \sin x = x^2,$$

by Gauß's theorem that the flux of \mathbf{V} through ∂T is given by

$$(3) \quad \int_{\partial T} \mathbf{V} \cdot \mathbf{n} \, dS = \int_T \operatorname{div} \mathbf{V} \, dx \, dy \, dz = \int_T x^2 \, dx \, dy \, dz.$$

The bounds of the tetrahedron give the estimates

$$4 - x - 2y \leq z \leq 8 - 2x - 4y = 2(4 - x - 2y),$$

hence $4 - x - 2y \geq 0$, and thus $0 \leq x \leq 4 - 2y$ and $0 \leq y \leq 2$. By a reduction of (3) we then get

$$\begin{aligned} \int_{\partial T} \mathbf{V} \cdot \mathbf{n} \, dS &= \int_T x^2 \, dx \, dy \, dz = \int_0^2 \left\{ \int_0^{4-2y} \left(\int_{4-x-2y}^{8-2x-4y} 8 - 2x - 4yx^2 \, dz \right) dx \right\} dy \\ &= \int_0^2 \left\{ \int_0^{4-2y} x^2(4-x-2y) \, dx \right\} dy = \int_0^2 \left\{ \int_0^{4-2y} (4x^2 - x^3 - 2yx^2) \, dx \right\} dy \\ &= \int_0^2 \left[\frac{4}{3} x^3 - \frac{1}{4} x^4 - \frac{2}{3} yx^3 \right]_{x=0}^{4-2y} dy \\ &= \int_0^2 \left\{ \frac{4}{3} (2\{2-y\})^3 - \frac{1}{4} (2\{2-y\})^4 - \frac{2}{3} y (2\{2-y\})^3 \right\} dy \\ &= \int_0^2 \left\{ \frac{32}{3} (2-y)^3 - \frac{16}{4} (2-y)^4 - \frac{16}{3} y (2-y)^3 \right\} dy \\ &= \left(\frac{16}{3} - \frac{16}{4} \right) \int_0^2 (2-y)^4 \, dy = \frac{16}{12} \int_0^2 t^4 \, dt = \frac{4}{3} \left[\frac{1}{5} t^5 \right]_0^2 = \frac{128}{15}. \end{aligned}$$

Example 4.10 Given the vector field

$$\mathbf{V}(x, y, z) = (4x + 3y^3, 9xy^2 + z, y), \quad (x, y, z) \in \mathbb{R}^3.$$

- 1) Find $\operatorname{div} \mathbf{V}$ and $\operatorname{rot} \mathbf{V}$.
- 2) Show that \mathbf{V} is a gradient field and find all its integrals.
- 3) Compute the tangential line integral

$$\int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} \, ds = \int_{\mathcal{K}} (4x + 3y^3) \, dx + (9xy^2 + z) \, dy + y \, dz,$$

where \mathcal{K} denotes the line segment from the point $(0, 0, 0)$ to the point $(1, 1, 1)$.

- 4) Find the flux of \mathbf{V} through the unit sphere $x^2 + y^2 + z^2 = 1$ with a normal vector pointing away from the ball.

A Vector analysis.

D Follow the guidelines

I 1) We get by direct computations

$$\operatorname{div} \mathbf{V} = 4 + 18xy^2,$$

and

$$\operatorname{rot} \mathbf{V} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4x + 3y^3 & 9xy^2 + z & y \end{vmatrix} = (1 - 1, 0 - 0, 9y^2 - 9y^2) = (0, 0, 0),$$

and we note that \mathbf{V} is rotation free.

- 2) Since the field is rotation free and the domain is simply connected, we conclude that \mathbf{V} is a gradient field. Then by calculating the differential form,

$$\begin{aligned} \mathbf{V} \cdot (dx, dy, dz) &= (4x + 3y^3) \, dx + (9xy^2 + z) \, dy + y \, dz \\ &= 4x \, dx + 3(y^3 \, dx + x \cdot 3y^2 \, dy) + (z \, dy + y \, dz) \\ &= d(2x^2 + 3xy^3 + yz), \end{aligned}$$

and it follows once more that \mathbf{V} is a gradient field with all its integrals given by

$$F(x, y, z) = 2x^2 + 3xy^3 + yz + C, \quad C \in \mathbb{R}.$$

- 3) We have proved that \mathbf{V} is a gradient field with an integral F . Then it follows that

$$\begin{aligned} \int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} \, ds &= \int_{\mathcal{K}} (4x + 3y^3) \, dx + (9xy^2 + z) \, dy + y \, dz \\ &= [F(x, y, z)]_{(0,0,0)}^{(1,1,1)} = [2x^2 + 3xy^3 + yz]_{(0,0,0)}^{(1,1,1)} = 2 + 3 + 1 = 6. \end{aligned}$$

4) An application of Gauß's theorem gives

$$\int_{\partial\Omega} \mathbf{V} \cdot \mathbf{n} dS = \int_{\Omega} \operatorname{div} \mathbf{V} d\Omega = \int_{\Omega} (4 + 18xy^2) d\Omega = 4 \operatorname{vol}(\Omega) + 0 = 4 \cdot \frac{4\pi}{3} = \frac{16\pi}{3},$$

because $\int_{\Omega} 18xy^2 d\Omega = 0$ of symmetric reasons. The integrand is odd in x , and the body is symmetric with respect to the (Y, Z) -plane.

Example 4.11 A body of revolution L with the Z -axis as axis of rotation is given in semi polar coordinates (ϱ, φ, z) given by the inequalities

$$0 \leq \varphi \leq 2\pi, \quad -a \leq z \leq a, \quad 0 \leq \varrho \leq a - \frac{z^2}{a},$$

where $a \in \mathbb{R}_+$ is some given constant.

1. Compute the space integral

$$I = \int_L z^2 d\Omega.$$

Given the vector field

$$\mathbf{V}(x, y, z) = (\cos x, y \sin x, z^3), \quad (x, y, z) \in \mathbb{R}^3.$$

2. Find the flux

$$\int_{\partial L} \mathbf{V} \cdot \mathbf{n} dS,$$

where the unit normal vector \mathbf{n} is pointing away from the body.

A Space integral and flux in semi polar coordinates.

D Slice up the body; apply Gauß's theorem.

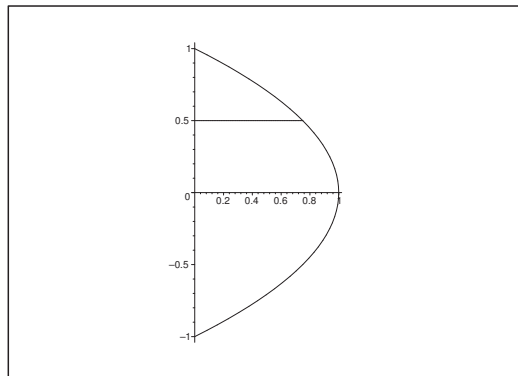


Figure 21: The meridian curve when $a = 1$.

I 1) It follows from the rearrangement

$$\varrho = a \left\{ 1 - \left(\frac{z}{a} \right)^2 \right\}$$

that the meridian curve is an arc of a parabola.

The space integral is computed by the method of slicing,

$$\begin{aligned} I &= \int_L z^2 d\Omega = \pi \int_{-a}^a \left(a - \frac{z^2}{a} \right)^2 z^2 dz = 2\pi \int_0^a \left(a^2 - 2z^2 + \frac{z^4}{a^2} \right) z^2 dz \\ &= 2\pi \int_0^a \left\{ a^2 z^2 - 2z^4 + \frac{z^6}{a^2} \right\} dz = 2\pi \left[\frac{a^2}{3} z^3 - \frac{2}{5} z^5 + \frac{z^7}{7a^2} \right]_0^a \\ &= 2\pi a^5 \left(\frac{1}{3} - \frac{2}{5} + \frac{1}{7} \right) = \frac{2\pi a^5}{105} (35 - 42 + 15) = \frac{16\pi a^5}{105}. \end{aligned}$$

2) The flux is according to Gauß's theorem given by

$$\begin{aligned} \int_{\partial L} \mathbf{V} \cdot \mathbf{n} dS &= \int_{\Omega} \operatorname{div} \mathbf{V} d\Omega = \int_{\Omega} \{ -\sin x + \sin x + sz^2 \} d\Omega \\ &= 3 \int_{\Omega} z^2 d\Omega = 3I = \frac{16\pi a^5}{35}, \end{aligned}$$

where we have used the result of 1).

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Example 4.12 Given the vector field

$$\mathbf{V}(x, y, z) = (3xz^2 - x^3, 3yz^2 - y^3, 3z(x^2 + y^2)), \quad (x, y, z) \in \mathbb{R}^3,$$

and the constant $a \in \mathbb{R}_+$.

1. Show that \mathbf{V} is a gradient field and find all its integrals.

Let \mathcal{K} be the curve which is composed of the quarter circle of centrum at $(0, 0, 0)$ and runs from $(a, 0, 0)$ to $(0, a, 0)$, and the line segment from $(0, a, 0)$ to $(0, a, 2a)$.

2. Find the tangential line integral

$$\int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} \, ds.$$

3. Find the flux of \mathbf{V} through the surface of the ball of centrum $(0, 0, 0)$ and radius a .

A Vector analysis.

D Each question can be answered in several ways. We shall here demonstrate some of the variants.

I 1) First note that \mathbf{V} is of class C^∞ .

First variant. Prove directly by some manipulation that the differential form $\mathbf{V} \cdot d\mathbf{x}$ can be written as dF where F then by the definition is an integral. Do this by pairing terms which are similar to each other.

$$\begin{aligned} \mathbf{V} \cdot d\mathbf{x} &= (3xz^2 - x^3)dx + (3yz^2 - y^3)dy + 3z(x^2 + y^2)dz \\ &= \frac{3}{2}z^2d(x^2) - \frac{1}{4}d(x^4) + \frac{3}{2}z^2d(y^2) - \frac{1}{4}d(y^4) + \frac{3}{2}(x^2 + y^2)d(z^2) \\ &= d\left(\frac{3}{2}(x^2 + y^2)z^2 - \frac{1}{4}x^4 - \frac{1}{4}y^4\right). \end{aligned}$$

It follows immediately from this that \mathbf{V} is a gradient field and that all integrals are given by

$$F(x, y, z) = \frac{3}{2}(x^2 + y^2)z^2 - \frac{1}{4}x^4 - \frac{1}{4}y^4 + C,$$

where C is an arbitrary constant.

Second variant. Clearly, \mathbb{R}^3 is simply connected. Furthermore,

$$\begin{aligned} \frac{\partial L}{\partial y} = 0, \quad \frac{\partial M}{\partial x} = 0, \quad \text{s\aa} \frac{\partial L}{\partial y} = \frac{\partial M}{\partial x}, \\ \frac{\partial L}{\partial z} = 6xz, \quad \frac{\partial N}{\partial x} = 6xz, \quad \text{s\aa} \frac{\partial L}{\partial z} = \frac{\partial N}{\partial x}, \\ \frac{\partial M}{\partial z} = 6yz, \quad \frac{\partial N}{\partial y} = 6yz, \quad \text{s\aa} \frac{\partial M}{\partial z} = \frac{\partial N}{\partial y}. \end{aligned}$$

Since all the “mixed derivatives” are equal, it follows that $\mathbf{V} \cdot d\mathbf{x}$ is closed and thus exact. This means that \mathbf{V} is a gradient field and the integrals of \mathbf{V} exist.

In this variant we shall find the integrals by using line integrals. There are two subvariants:

a) *Integration along the broken line*

$$(0, 0, 0) \longrightarrow (x, 0, 0) \longrightarrow (x, y, 0) \longrightarrow (x, y, z).$$

In this case,

$$\begin{aligned} F_0(x, y, z) &= \int_0^x (-t^3) dt + \int_0^y (-t^3) dt + \int_0^z 3t(x^2 + y^2) dt \\ &= \frac{3}{2}(x^2 + y^2)z^2 - \frac{1}{4}(x^4 + y^4). \end{aligned}$$

The integrals are

$$F(x, y, z) = \frac{3}{2}(x^2 + y^2)z^2 - \frac{1}{4}(x^4 + y^4) + C,$$

where C is an arbitrary constant.

b) *Radial integration along $(0, 0, 0) \longrightarrow (x, y, z)$.*

The coordinates of \mathbf{V} are homogeneous of degree 3. Hence,

$$\begin{aligned} F_0(x, y, z) &= (x, y, z) \cdot \left((3xz^2 - x^3) \int_0^1 t^3 dt, (3yz^2 - y^3) \int_0^1 t^3 dt, 3z(x^2 + y^2) \int_0^1 t^3 dt \right) \\ &= \frac{1}{4}(x, y, z) \cdot (3xz^2 - x^3, 3yz^2 - y^3, 3z(x^2 + y^2)) \\ &= \frac{1}{4} \{ 3x^2z^2 - x^4 + 3y^2z^2 - y^4 + 3z^2(x^2 + y^2) \} \\ &= \frac{3}{2}(x^2 + y^2)z^2 - \frac{1}{4}(x^4 + y^4). \end{aligned}$$

The integrals are

$$F(x, y, z) = \frac{3}{2}(x^2 + y^2)z^2 - \frac{1}{4}(x^4 + y^4) + C,$$

where C is an arbitrary constant.

Third variant. Start by one of the variants 2a) and 2b) above without proving in advance that \mathbf{V} is a gradient field. The *possible* candidates of the integrals are

$$F(x, y, z) = \frac{3}{2}(x^2 + y^2)z^2 - \frac{1}{4}(x^4 + y^4) + C.$$

Check these!:

$$\nabla F(x, y, z) = (3xz^2 - x^3, 3yz^2 - y^3, 3z(x^2 + y^2)) = \mathbf{V}(x, y, z).$$

This shows that \mathbf{V} is a gradient field and its integrals are given by

$$F(x, y, z) = \frac{3}{2}(x^2 + y^2)z^2 - \frac{1}{4}(x^4 + y^4) + C,$$

where C is an arbitrary constant.

Fourth variant. *Improper integration.*

First put

$$\omega = \mathbf{V} \cdot d\mathbf{x} = (3xz^2 - x^3)dx + (3yz^2 - y^3)dy + 3z(x^2 + y^2)dz.$$

By an improper integration of the first term on the right hand side we get

$$F_1(x, y, z) = \int^x (3tz^2 - t^3)dt = \frac{3}{2}x^2z^2 - \frac{1}{4}x^4.$$

The differential is

$$dF_1 = (3xz^2 - x^3)dx + 3x^2z dz,$$

hence

$$\omega - dF_1 = (3yz^2 - y^3)dy + 3zy^2 dz,$$

which neither contains x nor dx .

When we repeat this procedure on $\omega - dF_1$ we get

$$F_2(y, z) = \int^y (3tz^2 - t^3)dt = \frac{3}{2}y^2z^2 - \frac{1}{4}y^4$$

with the differential

$$dF_2 = (3yz^2 - y^3)dy + 3zy^2 dz = \omega - dF_1.$$

Then by a rearrangement,

$$\omega = \mathbf{V} \cdot d\mathbf{x} = dF_1 + dF_2 = d\left(\frac{3}{2}x^2z^2 - \frac{1}{4}x^4 + \frac{3}{2}y^2z^2 - \frac{1}{4}y^4\right),$$

proving that \mathbf{V} is a gradient field with the integrals

$$F(x, y, z) = \frac{3}{2}(x^2 + y^2)z^2 - \frac{1}{4}(x^4 + y^4) + C,$$

C being an arbitrary constant.

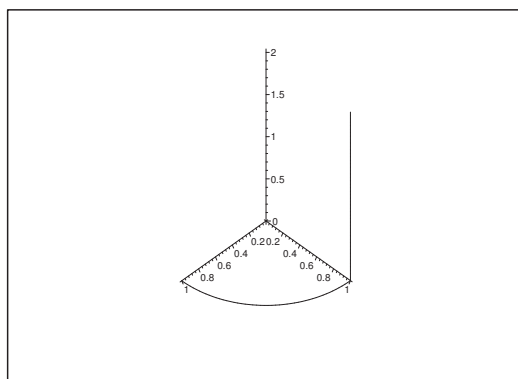


Figure 22: The curve \mathcal{K} for $a = 1$.

2) Here we have two variants.

First variant. Since \mathbf{V} is a gradient field with the integral

$$F_0(x, y, z) = \frac{3}{2}(x^2 + y^2)z^2 - \frac{1}{4}(x^4 + y^4),$$

and \mathcal{K} is a connected curve, we have

$$\begin{aligned} \int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} \, ds &= F_0(0, a, 2a) - F_0(a, 0, 0) \\ &= \frac{3}{2}(0^2 + a^2) \cdot 4a^2 - \frac{1}{4}(0^4 + a^4) + \frac{1}{4}(a^4 + 0^4) = 6a^4. \end{aligned}$$

Second variant. The definition of a tangential line integral.

The curve \mathcal{K} is composed of the two subcurves

$$\mathcal{K}_1 : (x(t), y(t), z(t)) = a(\cos t, \sin t, 0), \quad t \in \left[0, \frac{\pi}{2}\right],$$

$$\mathcal{K}_2 : (x(t), y(t), z(t)) = a(0, 1, t), \quad t \in [0, 2].$$

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First calculate

$$\begin{aligned}\int_{\mathcal{K}_1} \mathbf{V} \cdot \mathbf{t} \, ds &= \int_0^{\frac{\pi}{2}} a^3 (-\cos^3 t, -\sin^3 t, 0) \cdot a(-\sin t, \cos t, 0) \, dt \\ &= a^4 \int_0^{\frac{\pi}{2}} \{\cos^3 t \cdot \sin t - \sin^3 t \cdot \cos t\} \, dt \\ &= \frac{a^4}{4} [-\cos^4 t - \sin^4 t]_0^{\frac{\pi}{2}} = \frac{a^4}{4} \{-1 + 1\} = 0,\end{aligned}$$

and

$$\begin{aligned}\int_{\mathcal{K}_2} \mathbf{V} \cdot \mathbf{t} \, ds &= \int_0^2 a^3 (0, 3t^2 - 1, 3t(0^2 + 1^2)) \cdot a(0, 0, 1) \, dt \\ &= a^4 \int_0^2 3t \, dt = \frac{3}{2} a^4 \cdot 4 = 6a^4.\end{aligned}$$

Summarizing we get

$$\int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} \, ds = \int_{\mathcal{K}_1} \mathbf{V} \cdot \mathbf{t} \, ds + \int_{\mathcal{K}_2} \mathbf{V} \cdot \mathbf{t} \, ds = 0 + 6a^4 = 6a^4.$$

3) This problem can also be solved in various ways.

First variant. According to Gauß's theorem,

$$\text{flux} = \int_{\overline{K}(\mathbf{0};a)} \operatorname{div} \mathbf{V} \, d\Omega = \int_{\overline{K}(\mathbf{0};a)} 6z^2 \, d\Omega,$$

because

$$\operatorname{div} \mathbf{V} = 3z^2 - 3x^2 + 3z^2 - 3y^2 + 3(x^2 + y^2) = 6z^2.$$

The computation of this integral is most probably performed in one of the following sub-variants, although there exist some other (and more difficult) ways of calculation.

a) *Partition of $\overline{K}(\mathbf{0};a)$ into slices parallel to the XY -plane.*

By using this slicing method we get

$$\begin{aligned}\text{flux} &= \int_{\overline{K}(\mathbf{0};a)} 6z^2 \, d\Omega = \int_{-a}^a \left\{ \int_{\overline{K}((0,0);\sqrt{a^2-z^2})} 6z^2 \, dx \, dy \right\} dz \\ &= \int_{-a}^a 6z^2 \operatorname{area}(K(0,0);\sqrt{a^2-z^2}) \, dz = \int_{-a}^a 6z^2 \pi(a^2 - z^2) \, dz \\ &= 12\pi \int_0^a (a^2 z^2 - z^4) \, dz = 12\pi \left[\frac{1}{3} a^2 z^3 - \frac{1}{5} z^5 \right]_0^a = 12\pi a^5 \cdot \frac{2}{15} = \frac{8\pi}{5} a^5.\end{aligned}$$

b) *Computation in spherical coordinates:*

$$\begin{aligned}\text{flux} &= \int_{\overline{K}(\mathbf{0};a)} 6z^2 \, d\Omega = \int_0^{2\pi} \left\{ \int_0^\pi \left(\int_0^a 6r^2 \cos^2 \theta \cdot r^2 \sin \theta \, dr \right) d\theta \right\} d\varphi \\ &= 2\pi \int_0^\pi 6 \cos^2 \theta \cdot \sin \theta \, d\theta \cdot \int_0^a r^4 \, dr = 2\pi [2(-\cos^3 \theta)]_0^\pi \cdot \frac{a^5}{5} \\ &= \frac{4\pi}{5} a^5 (1 + 1) = \frac{8\pi}{5} a^5.\end{aligned}$$

Second variant. Direct application of the definition.

Put $\mathcal{F} = \partial K(\mathbf{0}; a)$. Then the unit normal vector field on \mathcal{F} is given by

$$\mathbf{n} = \frac{1}{a} (x, y, z).$$

By insertion into the definition,

$$\begin{aligned} \text{flux} &= \int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n} \, dS = \frac{1}{a} \int_{\mathcal{K}} \{3x^2z^2 - x^4 + 3y^2z^2 - y^4 + 3z^2(x^2 + y^2)\} \, dS \\ &= \frac{1}{a} \int_{\mathcal{F}} \{6z^2(x^2 + y^2) - x^4 - y^4\} \, dS. \end{aligned}$$

We shall in the following compute this surface integral in two different ways. Notice that there are many other possibilities. In both of these two subvariants we shall need the following:

Calculations:

$$\begin{aligned} (4) \quad & \int_0^{2\pi} (\cos^4 \varphi + \sin^4 \varphi) \, d\varphi \\ &= \int_0^{2\pi} (\cos^4 \varphi + \sin^4 \varphi + 2 \sin^2 \varphi \cos^2 \varphi - 2 \sin^2 \varphi \cos^2 \varphi) \, d\varphi \\ &= \int_0^{2\pi} \left\{ (\cos^2 \varphi + \sin^2 \varphi)^2 - \frac{1}{2} \sin^2 2\varphi \right\} \, d\varphi \\ &= \int_0^{2\pi} \left\{ 1 - \frac{1}{2} \cdot \frac{1}{2} (1 - \cos 4\varphi) \right\} \, d\varphi = \frac{3}{4} \cdot 2\pi = \frac{3\pi}{2}. \end{aligned}$$

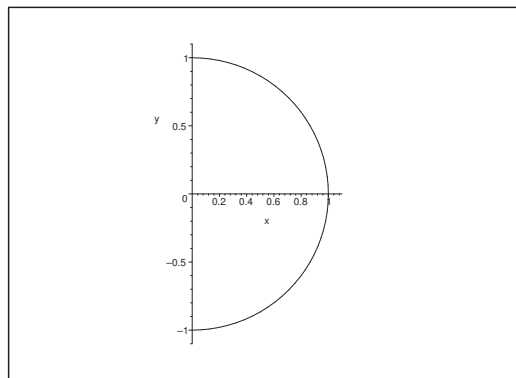


Figure 23: The meridian curve \mathcal{M} .

a) Consider the surface \mathcal{F} as a surface of revolution with the meridian curve

$$\mathcal{M}: \quad \varrho(z) = \sqrt{a^2 - z^2}, \quad z \in [-a, a],$$

thus

$$x(z) = \sqrt{a^2 - z^2} \cos \varphi, \quad y = \sqrt{a^2 - z^2} \sin \varphi, \quad z = z,$$

and the weight function

$$\sqrt{\{\varrho'(z)\}^2 + 1} = \sqrt{1 + \frac{z^2}{a^2 - z^2}} = \sqrt{\frac{a^2}{a^2 - z^2}} = \frac{a}{\sqrt{a^2 - z^2}}.$$

By insertion into a suitable formula we get

$$\begin{aligned}
 \text{flux} &= \frac{1}{a} \int_{\mathcal{F}} \{6z^2(x^2 + y^2) - x^4 - y^4\} dS \\
 &= \frac{1}{a} \int_{-a}^a \left\{ \int_0^{2\pi} \{6z^2([a^2 - z^2] \cos^2 \varphi + [a^2 - z^2] \sin^2 \varphi) \right. \\
 &\quad \left. - (a^2 - z^2)^2 (\cos^4 \varphi + \sin^4 \varphi) \} d\varphi \right\} \frac{\sqrt{a^2 - z^2} \cdot a}{\sqrt{a^2 - z^2}} dz \\
 &= \int_{-a}^a \left\{ \int_0^{2\pi} \{6z^2(a^2 - z^2) - (a^2 - z^2)^2 (\cos^4 \varphi + \sin^4 \varphi) \} d\varphi \right\} dz \\
 &= \int_{-a}^a \left\{ 2\pi \cdot 6z^2(a^2 - z^2) - \frac{3\pi}{2} (a^2 - z^2)^2 \right\} dz \quad (\text{ved (4)}) \\
 &= 12\pi \int_{-a}^a (a^2 z^2 - z^4) dz - \frac{3\pi}{2} \int_{-a}^a (a^4 - 2a^2 z^2 + z^4) dz \\
 &= 2 \cdot 12\pi \left[\frac{a^2}{3} z^3 - \frac{1}{5} z^5 \right]_0^a - 2 \cdot \frac{3\pi}{2} \left[a^4 z - \frac{2}{3} a^2 z^3 + \frac{1}{5} z^5 \right]_0^a \\
 &= 24\pi a^5 \cdot \frac{2}{15} - 3\pi a^5 \left(1 - \frac{2}{3} + \frac{1}{5} \right) = \pi a^5 \cdot \left(\frac{16}{5} - 1 - \frac{3}{5} \right) = \frac{8\pi a^5}{5}.
 \end{aligned}$$

b) ALTERNATIVELY it follows by the symmetry that the flux through

$$\mathcal{F}_+ = \{(x, y, z) \in \mathcal{F} \mid z \geq 0\}$$

is equal to the flux through $\mathcal{F} \setminus \mathcal{F}_+$, thus

$$\text{flux} = \frac{2}{a} \int_{\mathcal{F}_+} \{6z^2(x^2 + y^2) - x^4 - y^4\} dS.$$

The surface \mathcal{F}_+ is the graph of

$$z = \sqrt{a^2 - x^2 - y^2}, \quad (x, y) \in B = \{(x, y) \mid x^2 + y^2 \leq a^2\},$$

and the normal vector is

$$\mathbf{N}(x, y) = \left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right) = \left(\frac{x}{\sqrt{a^2 - x^2 - y^2}}, \frac{y}{\sqrt{a^2 - x^2 - y^2}}, 1 \right),$$

hence

$$\|\mathbf{N}(x, y)\| = \frac{a}{\sqrt{a^2 - x^2 - y^2}}.$$

Then by

- i) reduction of the surface integral to a plane integral,
- ii) reduction in polar coordinates,
- iii) application of the calculation (4),
- iv) the change of variable $t = \sqrt{a^2 - r^2}$, i.e.

$$r^2 = a^2 - t^2 \quad \text{og} \quad dt = -\frac{r}{\sqrt{a^2 - r^2}} dr,$$

we finally get

$$\begin{aligned}
 \text{flux} &= \frac{2}{a} \int_B \{6(a^2 - x^2 - y^2)(x^2 + y^2) - x^4 - y^4\} \frac{a}{\sqrt{a^2 - x^2 - y^2}} dx dy \\
 &= 2 \int_0^{2\pi} \left\{ \int_0^a \{6(a^2 - r^2)r^2 - r^4(\cos^4 \varphi + \sin^4 \varphi)\} \frac{r}{\sqrt{a^2 - r^2}} dr \right\} d\varphi \\
 &= 2 \int_0^a \left\{ 12\pi(a^2 - r^2)r^2 - \frac{3\pi}{2} r^4 \right\} \frac{r}{\sqrt{a^2 - r^2}} dr \quad (\text{by (4)}) \\
 &= \pi \int_0^a \{24t^2(a^2 - t^2) - 3(a^2 - t^2)^2\} dt \\
 &= \pi \int_0^a \{24a^2t^2 - 24t^4 - 3a^4 + 6a^2t^2 - 3t^4\} dt \\
 &= \pi a^5 \left\{ 8 - \frac{24}{5} - 3 + 2 - \frac{3}{5} \right\} = \pi a^5 \left\{ 7 - \frac{27}{5} \right\} \\
 &= \pi a^5 \cdot \frac{35 - 27}{5} = \frac{8\pi a^5}{5}.
 \end{aligned}$$

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Example 4.13 Let a be a positive constant. Consider the set

$$A = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq a^2, 0 \leq y, -y \leq x \leq y, |z| \leq 2a\}.$$

- 1) Describe A in semi polar coordinates (ϱ, φ, z) .
- 2) Compute the space integrals

$$I = \int_A x \, d\Omega, \quad J = \int_A y \, d\Omega, \quad K = \int_A z^2 \, d\Omega.$$

- 3) Find the flux of the vector field

$$\mathbf{V}(x, y, z) = (3xz^2 + \cosh y, z^2 e^x, z^3 - 3axz + \sinh y), \quad (x, y, z) \in \mathbb{R}^3,$$

through the surface ∂A with its normal vector pointing outwards.

A Space integrals; flux.

D The first two problems are solved by the reduction theorems. In 3) we apply Gauß's theorem.

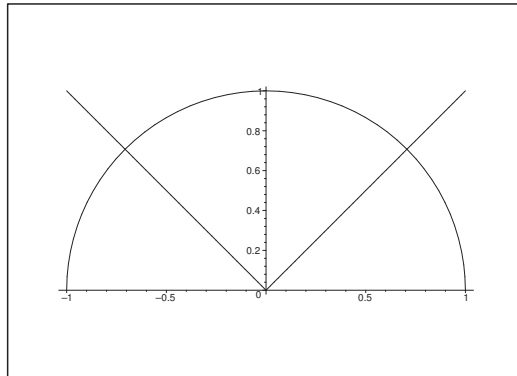


Figure 24: The domain B for $a = 1$ lies inside the upper angular space and inside the half circle.

- I** 1) Clearly, A is a cylinder with a quarter disc B in the (X, Y) -plane as generating surface. Thus A is described in semi polar coordinates by

$$A = \left\{ (\varrho, \varphi, z) \mid 0 \leq \varrho \leq a, \frac{\pi}{4} \leq \varphi \leq \frac{3\pi}{4}, -2a \leq z \leq 2a \right\}.$$

- 2) By an argument of symmetry (first integrate with respect to x) we get

$$I = \int_A x \, d\Omega = 0.$$

ALTERNATIVELY,

$$\begin{aligned} I &= \int_A x \, d\Omega = \int_{-2a}^{2a} \left\{ \int_0^a \left(\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \varrho \cos \varphi \cdot \varrho \, d\varphi \right) d\varrho \right\} dz \\ &= 4a \int_0^a \varrho^2 \, d\varrho \cdot \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cos \varphi \, d\varphi = 4a \cdot \frac{a^3}{3} [\sin \varphi]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = 0. \end{aligned}$$

Furthermore,

$$\begin{aligned} J &= \int_A y \, d\Omega = \int_{-2a}^{2a} \left\{ \int_0^a \left(\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \varrho \sin \varphi \cdot \varrho \, d\varphi \right) d\varrho \right\} dz \\ &= 4a \cdot \frac{a^3}{3} [-\cos \varphi]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = \frac{4a^4}{3} \cdot \left\{ \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right\} = \frac{4\sqrt{2}a^4}{3}. \end{aligned}$$

Finally, by the slicing method,

$$K = \int_A z^2 \, d\Omega = \int_{-2a}^{2a} z^2 \operatorname{areal}(B) \, dz = \frac{1}{4} \cdot \pi a^2 \left[\frac{z^3}{3} \right]_{-2a}^{2a} = \frac{1}{4} \pi a^2 \cdot 2 \cdot \frac{8a^3}{3} = \frac{4\pi a^5}{3}.$$

3) By an application of Gauß's theorem,

$$\operatorname{flux} = \int_A \operatorname{div} \mathbf{V} \, d\Omega = \int_A \{3z^2 + 0 + 3z^2 - 3az\} \, d\Omega = 6K - 3aI = 8\pi a^5,$$

where we have inserted the values of K and I found in 2).

Example 4.14 Consider the function

$$F(x, y, z) = x^4 + x e^y \sin z, \quad (x, y, z) \in \mathbb{R}^3,$$

and the vector field $\mathbf{V} = \nabla F$.

- 1) Find the divergence $\nabla \cdot \mathbf{V}$ and the rotation $\nabla \times \mathbf{V}$.
- 2) Check if \mathbf{V} has a vector potential.
- 3) Find the flux of \mathbf{V} through ∂A , where A is the half ball given by the inequalities

$$x^2 + y^2 + z^2 \leq 9, \quad z \leq 0.$$

- 4) Find the flux of \mathbf{V} through the surface \mathcal{F} given by

$$x^2 + y^2 + z^2 = 9, \quad z \leq 0.$$

Show the orientation of \mathcal{F} on a figure. (Hint: Use that the surface \mathcal{F} is a subset of the surface ∂A of 3).

A Divergence, rotation, flux.

D Find \mathbf{V} . Use the rules of calculations and finally also Gauß's theorem.

I 1) First compute

$$\mathbf{V} = \nabla F = (4x^3 + e^y \sin z, x e^y \sin z, x e^y \cos z).$$

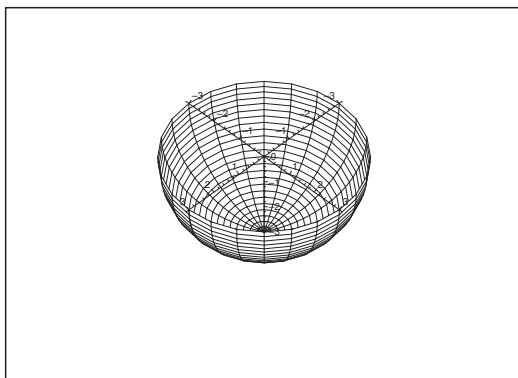
Then

$$\nabla \cdot \mathbf{V} = \nabla \cdot \nabla F = \Delta F = 12x^2 + x e^y \sin z - x e^y \sin z = 12x^2$$

and

$$\nabla \times \mathbf{V} = \nabla \times \nabla F = \mathbf{0},$$

which is obvious because \mathbf{V} is a gradient field and thence rotation free.

Figure 25: The body A .

- 2) Since \mathbf{V} is not divergence free in any open domain, \mathbf{V} does not have a vector potential.
 3) We get by Gauß's theorem, an argument of symmetry and using spherical coordinates,

$$\begin{aligned}
 \text{flux}(\partial A) &= \int_{\partial A} \mathbf{V} \cdot \mathbf{n} \, dS = \int_A \nabla \cdot \mathbf{V} \, d\Omega = 12 \int_A x^2 \, d\Omega = 12 \int_A y^2 \, d\Omega = 6 \int_A (x^2 + y^2) \, d\Omega \\
 &= 6 \int_0^{2\pi} \left\{ \int_{\frac{\pi}{2}}^{\pi} \left(\int_0^3 r^2 \sin^2 \theta \cdot r^2 \sin \theta \, dr \right) d\theta \right\} d\varphi \\
 &= 6 \cdot 2\pi \int_{\frac{\pi}{2}}^{\pi} (1 - \cos^2 \theta) \sin \theta \, d\theta \cdot \int_0^3 r^4 \, dr \\
 &= \frac{12\pi}{5} \cdot 3^5 \cdot \left[-\cos \theta + \frac{1}{3} \cos^3 \theta \right]_{\frac{\pi}{2}}^{\pi} = \frac{12\pi}{5} \cdot 3^5 \cdot \frac{2}{3} = \frac{1944\pi}{5}.
 \end{aligned}$$

- 4) Let \mathcal{G} denote the disc in the (X, Y) -plane with the unit normal vector field pointing upwards, and let \mathcal{F} denote the half sphere with the unit normal vector field pointing downward. Then according to 3),

$$\text{flux}(\partial A) = \text{flux}(\mathcal{F}) + \text{flux}(\mathcal{G}) = \frac{1944\pi}{5}.$$

Since $\mathbf{n} = (0, 0, 1)$ on \mathcal{G} , it follows by a rearrangement that

$$\begin{aligned}
 \text{flux}(\mathcal{F}) &= \frac{1944\pi}{5} - \text{flux}(\mathcal{G}) = \frac{1944\pi}{5} - \int_{\mathcal{G}} [xe^y \cos z]_{z=0} \, dS = \frac{1944\pi}{5} - \int_{\mathcal{G}} xe^y \, dS \\
 &= \frac{1944\pi}{5} - \int_{-3}^3 e^y \left\{ \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} x \, dx \right\} dy = \frac{1944\pi}{5} - 0 = \frac{1944\pi}{5},
 \end{aligned}$$

where we for symmetric reasons compute the plane integral over the disc in rectangular coordinates.

Example 4.15 The set $\Omega \subset \mathbb{R}^3$ is given in semi polar coordinates (ρ, φ, z) by the inequalities

$$-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}, \quad 0 \leq z \leq h, \quad 0 \leq \rho \leq a \left(1 - \frac{z}{h}\right),$$

where a and h are positive constants.

Also given the vector field

$$\mathbf{U}(x, y, z) = (x^3 z + 2y \cos x, y^3 z + y^2 \sin x, x^2 y^2), \quad (x, y, z) \in \mathbb{R}^3.$$

1) Find the divergence $\nabla \cdot \mathbf{U}$.

2) Find the flux Φ of the vector field \mathbf{U} through the surface $\partial\Omega$.

A Vector field, flux.

D Sketch a figure. Apply Gauß's theorem.

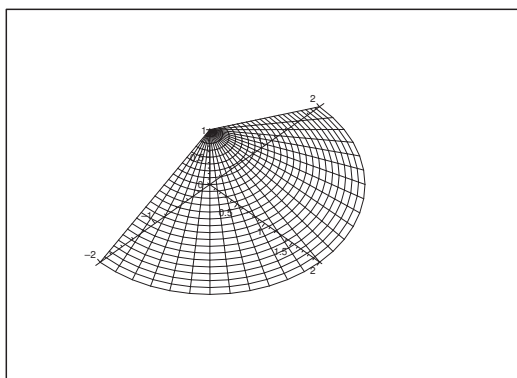


Figure 26: The body Ω for $a = 2$ and $h = 1$.

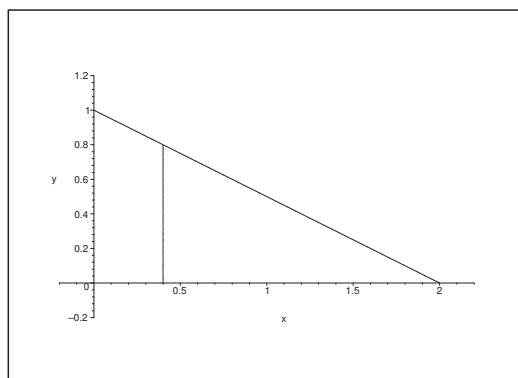


Figure 27: The meridian cut of Ω for $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $a = 2$, $h = 1$.

I We see that Ω is (half of) a cone (of revolution) with the top point $(0, 0, h)$ and a half disc in the (X, Y) -plane as its basis.

1) The divergence is

$$\operatorname{div} \mathbf{U} = \nabla \cdot \mathbf{U} = (3x^2z - 2y \sin x) + (3y^2z + 2y \sin x) + 0 = 3z(x^2 + y^2).$$

2) By applying Gauß's theorem and reducing in semi polar coordinates we conclude that the flux is

$$\begin{aligned} \Phi &= \int_{\Omega} \operatorname{div} \mathbf{U} \, d\Omega = \int_{\Omega} 3z(x^2 + y^2) \, d\Omega = \int_0^h \left\{ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\int_0^{a(1-\frac{z}{h})} 3z \varrho^2 \cdot \varrho \, d\varrho \right) d\varphi \right\} dz \\ &= 3\pi \int_0^h z \left(\int_0^{a(1-\frac{z}{h})} \varrho^3 \, d\varrho \right) dz = 3\pi \cdot \frac{1}{120} a^4 h^2 = \frac{\pi}{40} a^4 h^2. \end{aligned}$$

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An ALTERNATIVE computation is

$$\begin{aligned}\Phi &= \int_0^h \pi \cdot 3z \cdot \frac{1}{4} a^4 \left(1 - \frac{z}{h}\right)^4 dz = \frac{3\pi}{4} a^4 h \int_0^h \left\{1 - \left(1 - \frac{z}{h}\right)\right\} \left(1 - \frac{z}{h}\right)^4 dz \\ &= \frac{3\pi}{4} a^4 h \int_0^h \left\{\left(1 - \frac{z}{h}\right)^4 - \left(1 - \frac{z}{h}\right)^5\right\} dz = \frac{3\pi}{4} a^4 h^2 \int_0^1 \{\zeta^4 - \zeta^5\} d\zeta \\ &= \frac{3\pi}{4} a^4 h^2 \cdot \left(\frac{1}{5} - \frac{1}{6}\right) = \frac{\pi}{40} a^4 h^2.\end{aligned}$$

Example 4.16 Find the divergence and the rotation of the vector field

$$\mathbf{V}(x, y, z) = \left(2x + xy, 7x - \frac{1}{2}y^2, 3z\right), \quad (x, y, z) \in \mathbb{R}^3,$$

and find the flux of \mathbf{V} through the unit sphere $x^2 + y^2 + z^2 = 1$, where the normal vector is pointing outwards.

A Divergence, rotation and flux).

D Apply Gauß's theorem.

I The divergence is

$$\operatorname{div} \mathbf{V} = 2 + y - y + 3 = 5.$$

The rotation is

$$\operatorname{rot} \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x + xy & 7x - \frac{1}{2}y^2 & 3z \end{vmatrix} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 7x & 0 \end{vmatrix} = (0, 0, 7 - x).$$

By Gauß's theorem the flux through the surface \mathcal{F} of the unit sphere is given by

$$\int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n} dS = \int_{\Omega} \operatorname{div} \mathbf{V} d\Omega = \int_{\Omega} 5 d\Omega = 5 \operatorname{vol}(\Omega) = 5 \cdot \frac{4\pi}{4} \cdot 1^3 = \frac{20\pi}{3}.$$

Example 4.17 .

1) Find the volume of the body of revolution

$$A = \left\{ (x, y, z) \in \mathbb{R}^3 \mid \frac{1}{2}x^2 + \frac{1}{2}y^2 - 1 \leq z \leq 1 \right\}.$$

2) Find the flux of the vector field

$$\mathbf{V}(x, y, z) = (y^2 + x, xz^2 - yx^2, x^2z), \quad (x, y, z) \in \mathbb{R}^3,$$

through ∂A , where the unit normal vector is always pointing away from the body.

A Volume and flux.

D Sketch a section of A in the meridian half plane. Apply the method of slicing by finding the volume. The flux is found by means of Gauß's theorem.

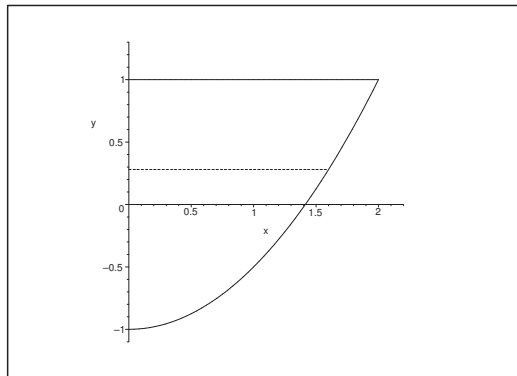


Figure 28: The meridian cut for A . The boundary curve has the equation $z = \frac{1}{2}\varrho^2 - 1$.

I 1) It follows from the sketch of the meridian half plane that the domain is described in semi polar coordinates by

$$0 \leq \varrho \leq \sqrt{2z+2}, \quad -1 \leq z \leq 1,$$

and that the body of revolution is a subset of a paraboloid of revolution.

THE SLICING METHOD. The paraboloid of revolution is intersected by a plane at the height $z \in]-1, 1]$ (the dotted line on the figure) in a circle of area

$$\pi \cdot \varrho(z)^2 = 2\pi(z+1).$$

Thus the volume of the body of revolution is

$$\text{vol}(A) = \int_{-1}^1 2\pi(z+1) dz = [\pi(z+1)^2]_{-1}^1 = 4\pi.$$

2) According to Gauss's theorem, the flux of \mathbf{V} through ∂A is given by

$$\int_{\partial A} \mathbf{V} \cdot \mathbf{n} dS = \int_A \text{div } \mathbf{V} d\Omega = \int_A \{1 - x^2 + x^2\} d\Omega = \text{vol}(A) = 4\pi.$$

5 Stokes's theorem

Example 5.1 Apply in each of the following cases Stokes's theorem to find the circulation of the given vector field $\mathbf{V} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ along the given closed curve \mathcal{K} , we one shall indicate the direction of the curve on a figure.

1) The circulation of the vector field

$$\mathbf{V}(x, y, z) = (y \sinh(xy) + z^2, x \sinh(xy) + z^2 + x, 2x^2 + 2y^2)$$

along the closed curve \mathcal{K} given by $x^2 + y^2 = 1, z = 1$.

2) The circulation of the vector field

$$\mathbf{V}(x, y, z) = (y^2 + z^4, (x - a)^2 + z^4, x^2 + y^2)$$

along the closed curve \mathcal{K} given by

$$x^2 + y^2 = bx, \quad z^4 = a^2 - x^2 - y^2, \quad \text{where } b < a \text{ and } z > 0.$$

3) The circulation of the vector field

$$\mathbf{V}(x, y, z) = (y, x - yz, x^2)$$

along the closed curve \mathcal{K} given by

$$x^2 + y^2 = 1, \quad z = 4 - 2x^2 - y^2.$$

4) The circulation of the vector field

$$\mathbf{V}(x, y, z) = (yz - 2y, xz + 4x, xy)$$

along the closed curve \mathcal{K} given by

$$\varrho = 1 + \cos \varphi, \quad z = \sqrt{4 - \varrho^2} \quad \text{for } \varphi \in [-\pi, \pi].$$

5) The circulation of the vector field

$$\mathbf{V}(x, y, z) = (y^2 - 2xy, 2xy, 2az + 3a^2)$$

along the closed curve \mathcal{K} given by

$$x^2 + y^2 = ax \quad z = a - \sqrt{x^2 + y^2}.$$

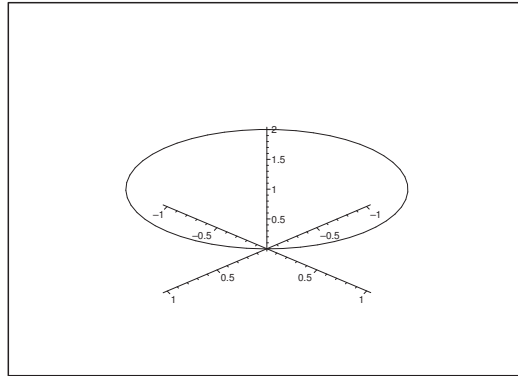
6) The circulation of the vector field $\mathbf{V}(x, y, z) = (z, x, y)$ along the boundary of the triangle \mathcal{K} of vertices $(0, 0, 1)$, $(0, 1, 0)$ and $(1, 0, 0)$.

7) The circulation of the vector field $\mathbf{V}(x, y, z) = (y, z, x)$ along the closed curve \mathcal{K} given by

$$x^2 + y^2 + z^2 = a^2, \quad z = y - x.$$

8) The circulation of the vector field $\mathbf{V}(x, y, z) = (y + \sin z, x, x \cos z)$ along the closed curve \mathcal{K} given by

$$x^2 + y^2 + z^2 = 1, \quad z = x.$$

Figure 29: The curve \mathcal{K} of **Example 5.1.1**.

- 9) The circulation of the vector field $\mathbf{V}(x, y, z) = (z^2, ax + z^2, 2x^2 + 2y^2)$ along the closed curve \mathcal{K} given by

$$x^2 + y^2 = a^2, \quad z = a.$$

- 10) The circulation of the vector field

$$\mathbf{V}(x, y, z) = (-y(x^2 + 2z^2), x(x^2 + 4y^2 + 2z^2), z^3)$$

along the closed curve \mathcal{K} given by

$$x^2 + y^2 = a^2, \quad z = a.$$

A Circulation of vector fields.

D Sketch the curve and choose a direction of it. Compute $\mathbf{rot} \mathbf{V}$, and choose the surface \mathcal{F} . Finally, apply Stokes's theorem.

I 1) The most obvious choice of the surface is

$$\mathcal{F} = \{(x, y, 1) \mid x^2 + y^2 \leq 1\}$$

where the orientation is given by the normal vector $\mathbf{n} = (0, 0, 1)$. Hence

$$\mathbf{n} \cdot \mathbf{rot} \mathbf{V} = \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} = \sinh(xy) + xy \cosh(xy) + 1 - \sinh(xy) - xy \cosh(xy) = 1.$$

According to Stokes's theorem the circulation is then

$$\int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} \, ds = \int_{\mathcal{F}} \mathbf{n} \cdot \mathbf{rot} \mathbf{V} \, dS = 1 \cdot \text{area}(\mathcal{F}) = \pi.$$

- 2) If we choose the orientation of \mathcal{K} , such that the projection of the curve onto the XY -plane has a positive direction, it is quite natural to choose the corresponding surface

$$\mathcal{F} = \left\{ (x, y, \sqrt{z^2 - x^2 - y^2}) \mid x^2 + y^2 \leq bx \right\},$$

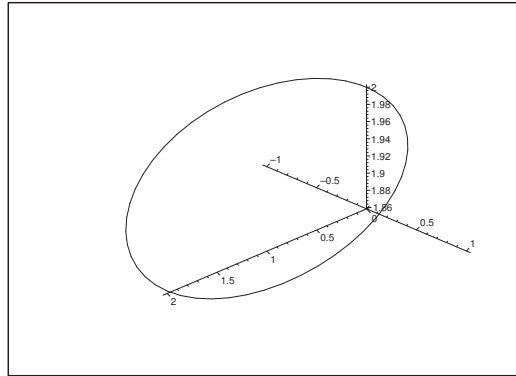


Figure 30: The curve \mathcal{K} of **Example 5.1.2** for $a = 4$ and $b = 2$.

with the normal vector $\mathbf{n} \cdot \mathbf{e}_z > 0$. We first find

$$\text{rot } \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 + z^4 & (x - a)^2 + z^4 & x^2 + y^2 \end{vmatrix} = (2y - 4z^3, 4z^3 - 2x, 2(x - a) - 2y).$$

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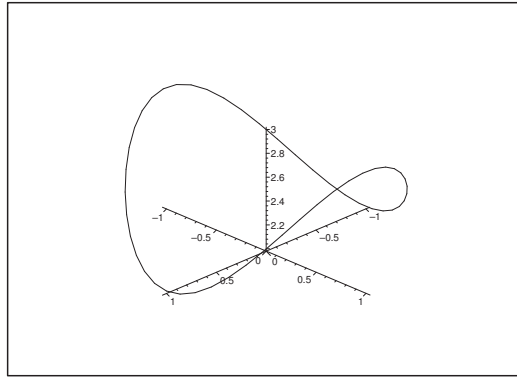
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Figure 31: The curve \mathcal{K} of **Example 5.1.3**

Then calculate the normal vector of the surface \mathcal{F} ,

$$\begin{aligned} \mathbf{N}(x, y) &= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 1 & 0 & -\frac{1}{2}x(\sqrt{a^2 - x^2 - y^2})^{-3} \\ 0 & 1 & -\frac{1}{2}y(\sqrt{a^2 - x^2 - y^2})^{-3} \end{vmatrix} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 1 & 0 & -\frac{1}{2}\frac{x}{z^3} \\ 0 & 1 & -\frac{1}{2}\frac{y}{z^3} \end{vmatrix} \\ &= \left(\frac{1}{2}\frac{x}{z^3}, \frac{1}{2}\frac{y}{z^3}, 1 \right), \end{aligned}$$

hence

$$\begin{aligned} \mathbf{n} \cdot \mathbf{rot} \mathbf{V} &= \frac{1}{\|\mathbf{N}(x, y)\|} \left\{ \frac{1}{2}\frac{x}{z^3}(2y - 4z^3) + \frac{1}{2}\frac{y}{z^3}(4z^3 - 2x) + 2(x - a) - 2y \right\} \\ &= \frac{1}{\|\mathbf{N}(x, y)\|} \left\{ \frac{xy}{z^3} - 2x + 2y - \frac{yx}{z^3} + 2x - 2a - 2y \right\} = -\frac{2a}{\|\mathbf{N}(x, y)\|}. \end{aligned}$$

Choose the parameter domain

$$B = \{(x, y) \mid x^2 + y^2 \leq bx\}.$$

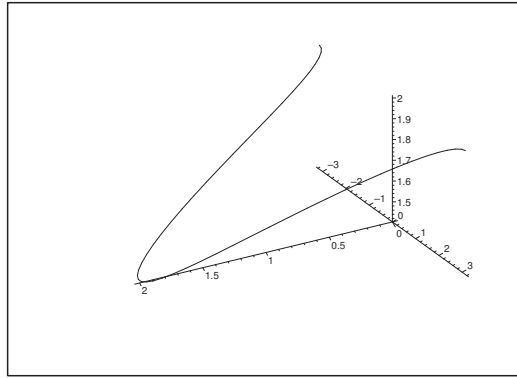
According to Stokes's theorem the circulation of \mathbf{V} along the curve \mathcal{K} is given by

$$\begin{aligned} \int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} \, ds &= \int_{\mathcal{F}} \mathbf{n} \cdot \mathbf{rot} \mathbf{V} \, dS = \int_B \frac{-2a}{\|\mathbf{N}(x, y)\|} \|\mathbf{N}(x, y)\| \, dx \, dy \\ &= -2a \, \text{area}(B) = -2a \cdot \pi \left(\frac{b}{2} \right)^2 = -\frac{\pi}{2} ab^2. \end{aligned}$$

3) Here we choose the surface

$$\mathcal{F} = \{(x, y, 2 - 2x^2 - y^2) \mid x^2 + y^2 \leq 1\},$$

where the boundary curve \mathcal{K} is oriented such that it is positive in the XY -plane. Then $\mathbf{n} \cdot \mathbf{e}_z > 0$ for the normal vector on \mathcal{F} .

Figure 32: The curve \mathcal{K} of **Example 5.1.4**

Then by computing,

$$\mathbf{rot} \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x - yz & x^2 \end{vmatrix} = (y, -2x, 1 - 1) = (y, -2x, 0).$$

and

$$\mathbf{N}(x, y) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 1 & 0 & -4x \\ 0 & 1 & -2y \end{vmatrix} = (4x, 2y, 1),$$

hence

$$\mathbf{n} \cdot \mathbf{rot} \mathbf{V} = \frac{1}{\|\mathbf{N}(x, y)\|} (2x, 2y, 1) \cdot (y, -2x, 0) = 0.$$

Then it is easy to find the circulation,

$$\int_{\mathcal{K}} \mathbf{t} \cdot \mathbf{V} ds = \int_{\mathcal{F}} \mathbf{n} \cdot \mathbf{rot} \mathbf{V} dS = 0.$$

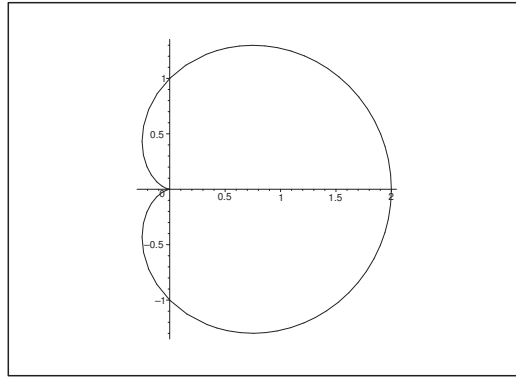
4) Choose the surface which is given in semi polar coordinates by

$$\mathcal{F} = \left\{ (\varrho, \varphi, \sqrt{4 - \varrho^2}) \mid 0 \leq \varrho \leq 1 + \cos \varphi, \varphi \in [-\pi, \pi] \right\} = \{(\varrho, \varphi, \sqrt{4 - \varrho^2}) \mid (\varrho, \varphi) \in B\},$$

where the parameter domain

$$B = \{(\varrho, \varphi) \mid 0 \leq \varrho \leq 1 + \cos \varphi, \varphi \in [-\pi, \pi]\}$$

lies inside the cardioid.

Figure 33: The parameter domain B of **Example 5.1.4**.

Choose the orientation of \mathcal{K} such that the projection of \mathcal{K} onto the cardioid is run through in the the positive sense of the plane. Then by a computation,

$$\mathbf{rot} \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz - 2y & xz + 4x & xy \end{vmatrix} = (x - x, y - y, z + 4 - z + 2) = (0, 0, 6).$$

The surface \mathcal{F} is described in rectangular coordinates (though in polar parameters) by

$$(x, y, z) = (\varrho \cos \varphi, \varrho \sin \varphi, \sqrt{4 - \varrho^2}).$$

This rectangular description is necessary when we compute the normal vector by the usual method,

$$\mathbf{N}(\varrho, \varphi) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \cos \varphi & \sin \varphi & -\frac{\varrho}{\sqrt{4 - \varrho^2}} \\ -\varrho \sin \varphi & \varrho \cos \varphi & 0 \end{vmatrix} = \left(\frac{\varrho^2 \cos \varphi}{\sqrt{4 - \varrho^2}}, \frac{\varrho^2 \sin \varphi}{\sqrt{4 - \varrho^2}}, \varrho \right),$$

thus

$$\mathbf{n} \cdot \mathbf{rot} \mathbf{V} = \frac{6\varrho}{\|\mathbf{N}(\varrho, \varphi)\|}.$$

The circulation along \mathcal{K} is

$$\begin{aligned} \int_{\mathcal{F}} \frac{6\varrho}{\|\mathbf{N}(\varrho, \varphi)\|} dS &= \int_B 6\varrho d\varrho d\varphi = \int_{-\pi}^{\pi} \left\{ \int_0^{1+\cos \varphi} 6\varrho d\varrho \right\} d\varphi \\ &= \int_{-\pi}^{\pi} 3(1 + \cos \varphi)^2 d\varphi = 3 \int_{-\pi}^{\pi} (1 + 2\cos \varphi + \cos^2 \varphi) d\varphi \\ &= 3 \int_{-\pi}^{\pi} \{1 + \cos^2 \varphi\} d\varphi + 0 = \frac{3}{2} \cdot 3 \cdot 2\pi = 9\pi. \end{aligned}$$

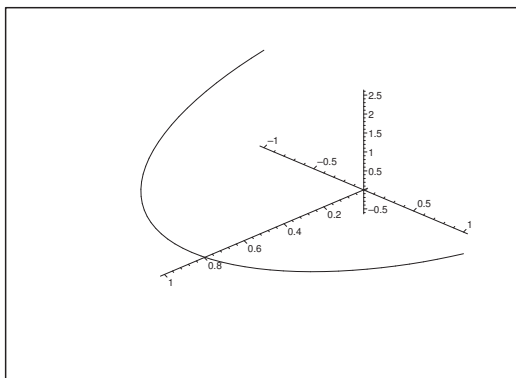


Figure 34: The curve \mathcal{K} of **Example 5.1.5** for $a = 1$.

5) Choose the surface

$$\mathcal{F} = \{(x, y, a - \sqrt{x^2 + y^2}) \mid x^2 + y^2 \leq ax\} = \{(x, y, a - \sqrt{x^2 + y^2}) \mid (x, y) \in B\},$$

where the parameter domain B is described in polar coordinates of the plane,

$$B = \left\{ (\varrho, \varphi) \mid \varrho \leq a \cos \varphi, \varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \right\}.$$

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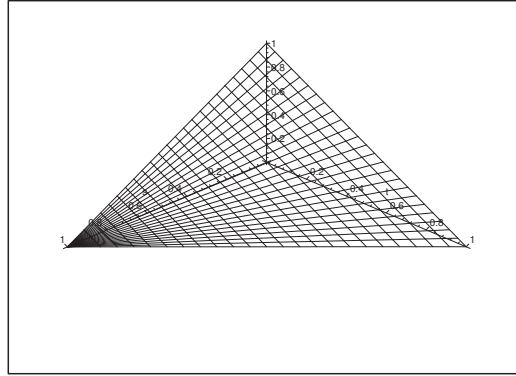
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Figure 35: The surface \mathcal{F} of Example 5.1.6.

Then

$$\mathbf{rot} \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - 2xy & 2xy & 2az + 3a^2 \end{vmatrix} = (0, 0, 2y - 2y + 2x) = (0, 0, 2x),$$

and

$$\mathbf{N}(x, y) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 1 & 0 & -\frac{x}{\sqrt{x^2 + y^2}} \\ 0 & 1 & -\frac{y}{\sqrt{x^2 + y^2}} \end{vmatrix} = \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}, 1 \right).$$

The circulation of \mathbf{V} along \mathcal{K} is then by Stokes's theorem,

$$\begin{aligned} \int_{\mathcal{K}} \mathbf{t} \cdot \mathbf{V} ds &= \int_{\mathcal{F}} \mathbf{n} \cdot \mathbf{V} dS = \int_B \mathbf{N} \cdot \mathbf{V} dx dy = \int_B 2x dx dy \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ \int_0^{a \cos \varphi} 2\rho \cos \varphi \cdot \rho d\rho \right\} d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{2}{3} \rho^3 \right]_0^{a \cos \varphi} \cos \varphi d\varphi \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2}{3} a^3 \cos^4 \varphi d\varphi = \frac{2}{3} a^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1 + \cos 2\varphi}{2} \right)^2 d\varphi \\ &= \frac{1}{6} a^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1 + 2 \cos 2\varphi + \frac{1 + \cos 4\varphi}{2} \right) d\varphi = \frac{1}{6} a^3 \cdot \frac{3}{2} \pi = \frac{\pi}{4} a^3. \end{aligned}$$

6) First calculate

$$\mathbf{rot} \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = (1, 1, 1).$$

We choose naturally the surface \mathcal{F} in the following way

$$\mathcal{F} = \{(x, y, 1 - x - y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1 - x\}$$

with the normal vector

$$\mathbf{N}(x, y) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = (1, 1, 1).$$

The circulation of the vector field along \mathcal{K} is then

$$\int_{\mathcal{K}} \mathbf{t} \cdot \mathbf{V} \, ds = \int_{\mathcal{F}} \mathbf{n} \cdot \mathbf{rot} \mathbf{V} \, dS = \int_B \mathbf{N} \cdot \mathbf{rot} \mathbf{V} \, dx \, dy = 3 \text{area}(B) = \frac{3}{2}.$$

7) Choose \mathcal{F} as the plane surface (a disc)

$$\mathcal{F} = \{(x, y, y - x) \mid x^2 + y^2 + (y - x)^2 \leq a^2\}$$

with $\mathbf{N} \cdot \mathbf{e}_z >$ by the chosen orientation. We get

$$\mathbf{rot} \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = (-1, -1, -1)$$

and

$$\mathbf{N}(x, y) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{vmatrix} = (1, -1, 1),$$

thus

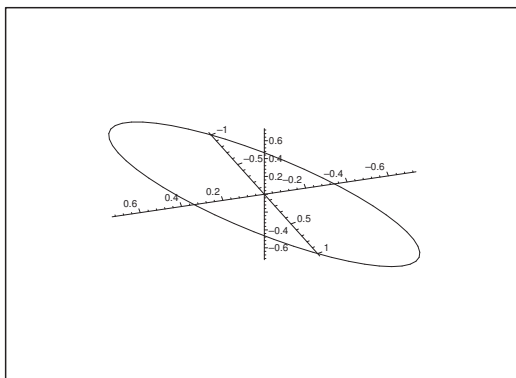
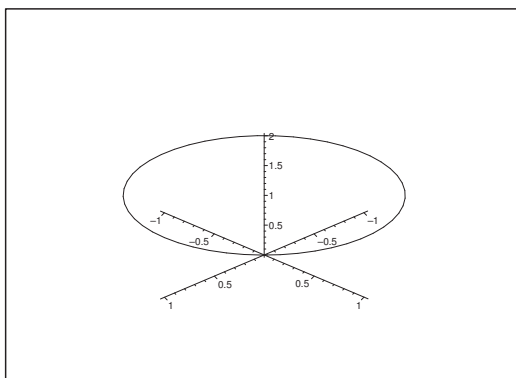
$$\mathbf{N} \cdot \mathbf{rot} \mathbf{V} = (-1, -1, -1) \cdot (1, -1, 1) = -1.$$

The projection B of \mathcal{F} onto the XY -plane is given by

$$a^2 \geq x^2 + y^2 + (y - x)^2 = x^2 + y^2 - \frac{1}{2}(x - y)^2 + \frac{3}{2}(x - y)^2 = \left(\frac{x + y}{\sqrt{2}}\right)^2 + 3\left(\frac{x - y}{\sqrt{2}}\right)^2,$$

which describes the interior of an elliptic disc with the directions of the axes

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \quad \text{and} \quad \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

Figure 36: The curve \mathcal{K} of **Example 5.1.8**.Figure 37: The curve \mathcal{K} of **Example 5.1.9** and **Example 5.1.10** for $a = 1$.

and the half axes a and $\frac{a}{\sqrt{3}}$.

The circulation is

$$\int_{\mathcal{K}} \mathbf{t} \cdot \mathbf{V} \, ds = \int_{\mathcal{F}} \mathbf{n} \cdot \mathbf{rot} \, \mathbf{V} \, dS = \int_B \mathbf{N} \cdot \mathbf{rot} \, \mathbf{V} \, dx \, dy = -\text{area}(B) = -\pi \cdot \frac{a^2}{\sqrt{3}}.$$

8) Since

$$\mathbf{rot} \, \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y + \sin z & x & x \cos z \end{vmatrix} = (0, \cos z - \cos z, 1 - 1) = (0, 0, 0),$$

the circulation is trivially 0 by Stokes's theorem.

9) Here

$$\text{rot } \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & ax + z^2 & 2x^2 + 2y^2 \end{vmatrix} = (4y - 2z, 2z - 4x, a).$$

We have $\mathbf{n} = (0, 0, 1)$ in the chosen orientation of \mathcal{K} , so the circulation becomes

$$\int_{\mathcal{K}} \mathbf{t} \cdot \mathbf{V} \, ds = \int_{\mathcal{F}} \mathbf{n} \cdot \text{rot } \mathbf{V} \, dS = \int_B a \, dx \, dy = a \cdot \text{area}(B) = \pi a^3.$$

ALTERNATIVELY it is here also easy to compute the circulation as a line integral. We choose the parametric description

$$(x, y, z) = (a \cos \varphi, a \sin \varphi, a), \quad \varphi \in [0, 2\pi],$$

for \mathcal{K} . Then we get the tangent vector field

$$\mathbf{t} \, ds = (-a \sin \varphi, a \cos \varphi, 0) d\varphi,$$

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hence

$$\begin{aligned}
 \int_{\mathcal{K}} \mathbf{t} \cdot \mathbf{V} \, ds &= \int_0^{2\pi} (-a \sin \varphi, a \cos \varphi, 0) \cdot (a^2, a^2 \cos \varphi + a^2, 2a^2) \, d\varphi \\
 &= a^3 \int_0^{2\pi} (-\sin \varphi + \cos^2 \varphi + \cos \varphi + 0) \, d\varphi \\
 &= a^3 \int_0^{2\pi} \cos^2 \varphi \, d\varphi = a^3 \int_0^{2\pi} \sin^2 \varphi \, d\varphi \\
 &= a^3 \int_0^{2\pi} \frac{\cos^2 \varphi + \sin^2 \varphi}{2} \, d\varphi = \frac{a^3}{2} \int_0^{2\pi} d\varphi = \pi a^3.
 \end{aligned}$$

10) The surface \mathcal{F} is the same as in **Example 5.1.9**, so we can reuse $\mathbf{n} = \mathbf{N} = (0, 0, 1)$ and

$$\mathbf{n} \cdot \mathbf{rot} \, \mathbf{V} = \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} = 3x^2 + 4y^2 + 2z^2 + x^2 + 2z^2 = 4(x^2 + y^2 + z^2).$$

Now $z = a$ on \mathcal{F} , so the circulation becomes

$$\begin{aligned}
 \int_{\mathcal{K}} \mathbf{t} \cdot \mathbf{V} \, ds &= \int_{\mathcal{F}} \mathbf{n} \cdot \mathbf{rot} \, \mathbf{V} \, dS = 4 \int_B (x^2 + y^2 + a^2) \, dx \, dy \\
 &= 4 \int_0^{2\pi} \left\{ \int_0^a \varrho^2 \cdot \varrho \, d\varrho \right\} d\varphi + 4a^2 \cdot \pi a^2 = 2\pi a^4 + 4\pi a^4 = 6\pi a^4.
 \end{aligned}$$

Example 5.2 Apply in each of the following cases Stokes's theorem to compute the flux

$$\int_{\mathcal{F}} \mathbf{n} \cdot \mathbf{rot} \, \mathbf{V} \, dS$$

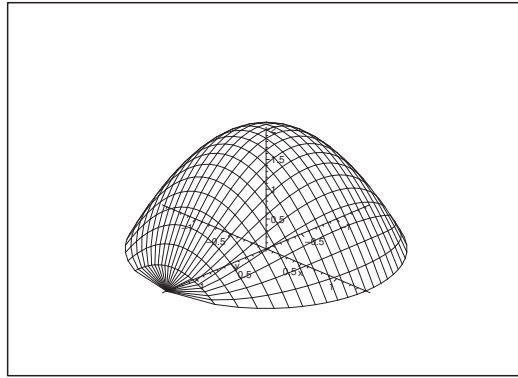
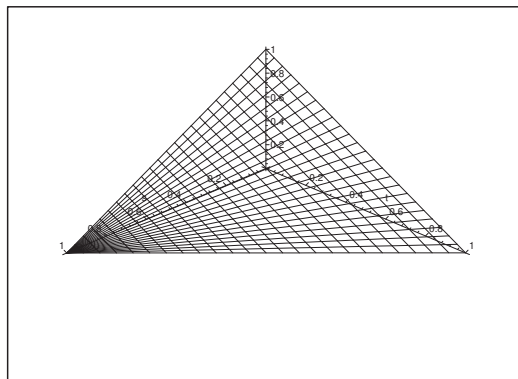
of the rotation of the given vector field $\mathbf{V} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ through the surface \mathcal{F} , where we shall choose an orientation, which is indicated on a figure.

- 1) The flux of $\mathbf{V}(x, y, z) = (y^2, x - 2xz, -xy)$ through the surface \mathcal{F} given by $z = \sqrt{a^2 - x^2 - y^2}$ for $x^2 + y^2 \leq a^2$.
- 2) The flux of $\mathbf{V}(x, y, z) = (2y^3, x^2 + yz, x)$ through the triangle \mathcal{F} with the vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.
- 3) The flux of $\mathbf{V}(x, y, z) = (y + z^2, z \ln(1 - x^2 + y^2), \text{Arctan}(xyz))$ through the surface \mathcal{F} given by $z = 1 - x^2 - y^2$ for $x^2 + y^2 \leq 1$.

A Flux computed by means of Stokes's theorem.

D Sketch the surface \mathcal{F} and the boundary curve \mathcal{K} and choose an orientation. (It has not been possible for me to sketch the orientation of the figures). Finally, exploit that the flux according to Stokes's theorem is given by

$$(5) \int_{\mathcal{F}} (\nabla \times \mathbf{V}) \cdot \mathbf{b} \, dS = \int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} \, ds, \quad \text{where } \mathcal{K} = \partial \mathcal{F}.$$

Figure 38: The surface \mathcal{F} of **Example 5.2.1** for $a = \sqrt{2}$.Figure 39: The surface \mathcal{F} of **Example 5.2.2**

- I 1) The boundary curve is the circle in the XY -plane of centrum $(0, 0)$ and radius a . Choose the parametric description

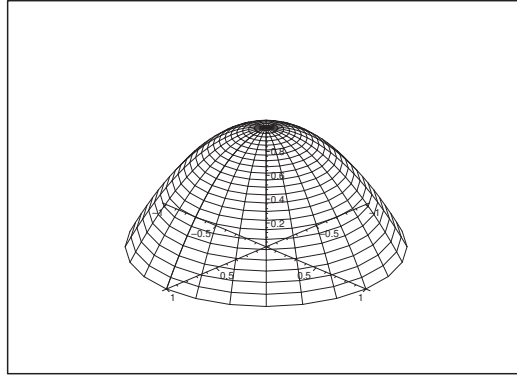
$$(x, y, z) = (a \cos \varphi, a \sin \varphi, 0), \quad \varphi \in [0, 2\pi],$$

for \mathcal{K} in \mathbb{R}^3 corresponding to a positive orientation. Since $\mathbf{t} = (-\sin \varphi, \cos \varphi, 0)$ and $ds = a d\varphi$, the flux is according to (5) given by

$$\begin{aligned} \int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} ds &= \int_0^{2\pi} (a^2 \sin^2 \varphi, a \cos \varphi, -a^2 \sin \varphi \cos \varphi) \cdot (-\sin \varphi, \cos \varphi, 0) a d\varphi \\ &= \int_0^{2\pi} \{-a^3 \sin^3 \varphi + a^2 \cos^2 \varphi + 0\} d\varphi = 0 + a^2 \cdot \frac{1}{2} \cdot 2\pi + 0 = \pi a^2. \end{aligned}$$

- 2) The boundary curve is the boundary of the triangle with e.g. the parametric description

$$\begin{cases} \mathcal{K}_1 : (x, y, z) = (1 - t, t, 0), & t \in [0, 1], \\ \mathcal{K}_2 : (x, y, z) = (0, 1 - t, t), & t \in [0, 1], \\ \mathcal{K}_3 : (x, y, z) = (t, 0, 1 - t), & t \in [0, 1], \end{cases}$$

Figure 40: The surface \mathcal{F} of **Example 5.2.3**.

where $ds = \sqrt{2} dt$ on each of the three subcurves. According to (5) the flux is given by

$$\begin{aligned}
 \int_{\mathcal{F}} (\nabla \times \mathbf{V}) \cdot \mathbf{n} dS &= \int_{\mathcal{K}_1} \mathbf{V} \cdot \mathbf{t} ds + \int_{\mathcal{K}_2} \mathbf{V} \cdot \mathbf{t} ds + \int_{\mathcal{K}_3} \mathbf{V} \cdot \mathbf{t} ds \\
 &= \int_0^1 (2t^3, (1-t)^2, 1-t) \cdot (-1, 1, 0) dt \\
 &\quad + \int_0^1 (2(1-t)^3, (1-t)t, 0) \cdot (0, -1, 1) dt \\
 &\quad + \int_0^1 (0, t^2, t) \cdot (1, 0, -1) dt \\
 &= \int_0^2 \{-2t^3 + (1-t)^2\} dt + \int_0^1 \{-(1-t)t\} dt + \int_0^1 (-t) dt \\
 &= \int_0^1 \{-2t^3 + (1-t)^2 + 1 - t^2 - (1-t) - t\} dt \\
 &= \int_0^1 \{-2t^3 + 2(t-1)^2 - 1\} dt = \left[-\frac{1}{2}t^4 + \frac{2}{3}(t-1)^3 - t \right]_0^1 \\
 &= -\frac{1}{2} - 1 + \frac{2}{3} = \frac{2}{3} - \frac{3}{2} = -\frac{5}{6}.
 \end{aligned}$$

- 3) The boundary curve \mathcal{K} is the unit circle in the XY -plane. Choose the orientation corresponding to the parametric description

$$(x, y, z) = (\cos \varphi, \sin \varphi, 0), \quad \varphi \in [0, 2\pi],$$

for \mathcal{K} . Then

$$\mathbf{t} = (-\sin \varphi, \cos \varphi, 0) \quad \text{and} \quad ds = d\varphi,$$

and the flux through \mathcal{F} is then according to (5.2) given by

$$\int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} ds = \int_0^{2\pi} (\sin \varphi, 0, 0) \cdot (-\sin \varphi, \cos \varphi, 0) d\varphi = - \int_0^{2\pi} \sin^2 \varphi d\varphi = -\frac{1}{2} \cdot 2\pi = -\pi.$$

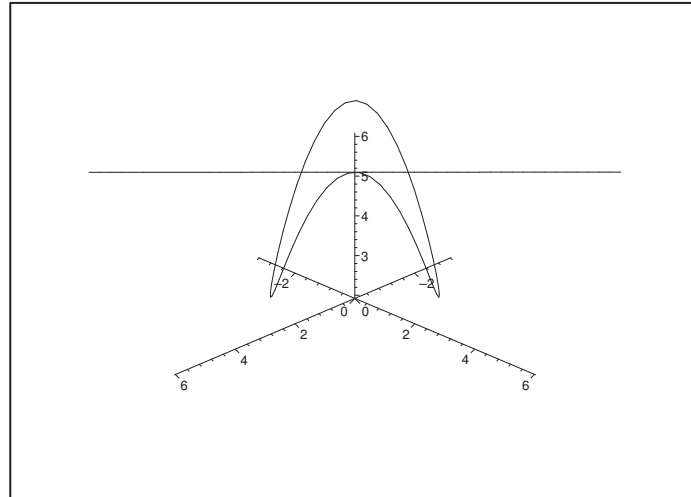


Figure 41: The space curve \mathcal{K} and its tangent at $\mathbf{r}\left(\frac{\pi}{4}\right)$.

Example 5.3 A space curve \mathcal{K} is given by the parametric description

$$\mathbf{r}(t) = (2 \cos t, 2 \sin t, 4 + 2 \sin(2t)), \quad t \in \mathbb{R}.$$

1. Find a parametric description of the tangent of \mathcal{K} at the point $\mathbf{r}\left(\frac{\pi}{4}\right)$.
2. Show that \mathcal{K} lies on the surface \mathcal{F} , given by the equation $z = 4 + xy$.

Let \mathcal{K}_1 be the restriction of \mathcal{K} corresponding to the parameter interval $[0, 2\pi]$, where this is run through corresponding to increasing t . Furthermore, we have given the vector field

$$\mathbf{V}(x, y, z) = (y, x, y^2 + 2z), \quad (x, y, z) \in \mathbb{R}^3.$$

3. Find the circulation of the vector field along the curve \mathcal{K}_1 .

A Space curve; circulation along a closed curve.

D Find $\mathbf{r}'(t)$ and the tangent corresponding to $t = \frac{\pi}{4}$.

Put $(x, y, z) = \mathbf{r}(t)$ into the equation of \mathcal{F} .

Try to apply Stokes's theorem. Alternatively, compute directly the circulation.

I 1) From

$$\mathbf{r}'(t) = (-2 \sin t, 2 \cos t, 4 \cos 2t)$$

follows that

$$\mathbf{r}\left(\frac{\pi}{4}\right) = \left(\frac{2}{\sqrt{2}}, \frac{2}{\sqrt{2}}, 4 + 2\right) = (\sqrt{2}, \sqrt{2}, 6)$$

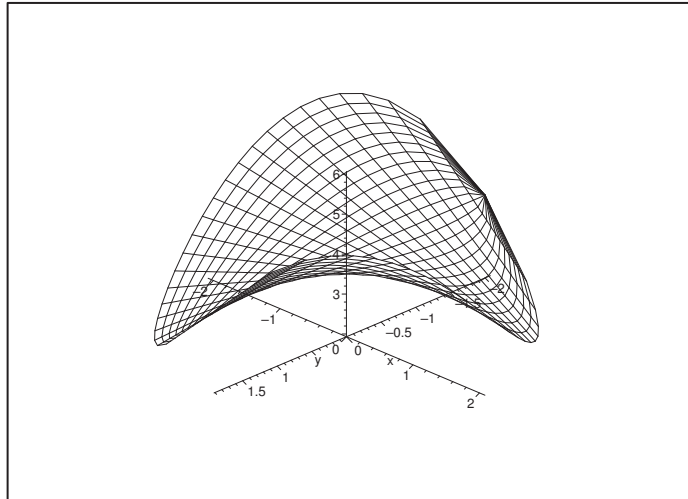


Figure 42: The surface \mathcal{F} of equation $z = 4 + xy$ for $x^2 + y^2 \leq 4$

and

$$\mathbf{r}'\left(\frac{\pi}{4}\right) = \left(-\frac{2}{\sqrt{2}}, \frac{2}{\sqrt{2}}, 0\right) = (-\sqrt{2}, \sqrt{2}, 0),$$

hence the equation of the tangent is

$$(x, y, z) = (\sqrt{2} - \sqrt{2}u, \sqrt{2} + \sqrt{2}u, 6) = (\sqrt{2}(1 - u), \sqrt{2}(1 + u), 6), \quad u \in \mathbb{R}.$$

2) Since

$$4 + x(t)y(t) = 4 + 4 \cos t \cdot \sin t = 4 + 2 \sin 2t = z,$$

the curve \mathcal{K} lies on the surface \mathcal{F} .

3) It follows from Stokes's theorem that

$$\int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} \, ds = \int_{\mathcal{F}} \mathbf{n} \cdot \mathbf{rot} \, \mathbf{V} \, dS,$$

where

$$\mathbf{rot} \, \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & y^2 + 2z \end{vmatrix} = (2y, -1, 0).$$

Since $z = 4 + xy$, $x^2 + y^2 \leq 4$, we get for the surface \mathcal{F} that

$$\frac{\partial z}{\partial x} = y \quad \text{and} \quad \frac{\partial z}{\partial y} = x,$$

thus

$$\mathbf{N}(x, y) = (-y, -x, 1).$$

ALTERNATIVELY, $(x, y, z) = (u, v, 4 + uv)$, thus

$$\mathbf{N}(u, v) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 1 & 0 & v \\ 0 & 1 & u \end{vmatrix} = (-v, -u, 1) = (-y, -x, 1).$$

If we put $B = \{(x, y) \mid x^2 + y^2 \leq 4\}$, then

$$\begin{aligned} \int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} \, ds &= \int_{\mathcal{F}} \mathbf{n} \cdot \mathbf{rot} \, \mathbf{V} \, dS = \int_B \mathbf{N} \cdot \mathbf{rot} \, \mathbf{V} \, dx \, dy \\ &= \int_B (-2y^2 + x + 0) \, dx \, dy = -2 \int_0^{2\pi} \left\{ \int_0^2 (\rho^2 \sin^2 \varphi + 0) \, \rho \, d\rho \right\} d\varphi \\ &= -2 \left[\frac{2\pi}{2} \right] \cdot \left[\frac{\rho^4}{4} \right]_0^2 = -2\pi \cdot \frac{16}{4} = -8\pi. \end{aligned}$$

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ALTERNATIVELY a direct computation gives

$$\begin{aligned}
 \int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} \, ds &= \int_0^{2\pi} \mathbf{V} \cdot \mathbf{r}'(t) \, dt \\
 &= \int_0^{2\pi} (2 \sin t, 2 \cos t, 4 \sin^2 t + 8 + 4 \sin 2t) \cdot (-\sin t, 2 \cos t, 4 \cos 2t) \, dt \\
 &= \int_0^{2\pi} \{-4 \sin^2 t + 4 \cos^2 t + 16 \sin^2 t \cos 2t + 32 \cos 2t + 16 \sin 2t \cos 2t\} \, dt \\
 &= 16 \int_0^{2\pi} \sin^2 t \cos 2t \, dt = 16 \int_0^{2\pi} \sin^2 t (2 \cos^2 t - 1) \, dt \\
 &= 8 \int_0^{2\pi} \sin^2 2t \, dt - 16\pi = 8 \cdot \frac{2\pi}{2} - 16\pi = -8\pi.
 \end{aligned}$$

Example 5.4 Let α be a constant, and consider the vector field

$$\mathbf{V}(x, y, z) = (\alpha x^2 + xz + yz, \alpha y^2 - xz - yz, \alpha(x^2 - y^2 + z^2)), \quad (x, y, z) \in \mathbb{R}^3.$$

1. Find $\operatorname{div} \mathbf{V}$.
2. Show that \mathbf{V} is not a gradient field in \mathbb{R}^3 for any choice of α .

Let \mathcal{K} denote the circle given by $x^2 + y^2 = a^2$, $z = a$.

3. Find the circulation of \mathbf{V} along \mathcal{K} ; indicate the chosen orientation.

Let the domain $\Omega \subset \mathbb{R}^3$ be given by $x^2 + y^2 \leq a^2$, $y \geq 0$, $0 \leq z \leq a$.

4. Find the flux of \mathbf{V} through $\partial\Omega$.

A Divergence, circulation and flux.

D Compute $\operatorname{div} \mathbf{V}$. Check $\frac{\partial V_i}{\partial x_j}$ for some i and j . Find the circulation, e.g. by Stokes's theorem.

Finally, apply Gauß's theorem to find the flux.

I 1) The divergence is

$$\operatorname{div} \mathbf{V} = 2\alpha x + z + 2\alpha y - z + 2\alpha z = 2\alpha(x + y + z).$$

2) It follows from

$$\frac{\partial V_1}{\partial y} = z \quad \text{and} \quad \frac{\partial V_2}{\partial z} = -z,$$

that

$$\frac{\partial V_1}{\partial y} \neq \frac{\partial V_2}{\partial x} \quad \text{for } z \neq 0.$$

The surface $z = 0$ does not contain inner points, thus \mathbf{V} is not a gradient field for any value of α .

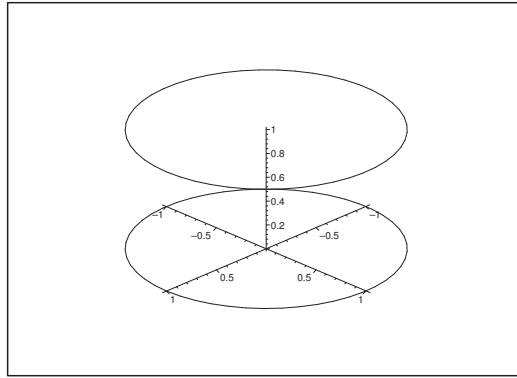


Figure 43: The curve \mathcal{K} and its projection onto the (X, Y) -plane for $a = 1$.

3) It follows by the definition of the circulation that

$$\begin{aligned}
 & \int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} \, ds \\
 &= \int_0^{2\pi} (\alpha a^2 \cos^2 t + a^2 \cos^2 t + a^2 \sin t, \alpha a^2 \sin^2 t - a^2 \cos t - a^2 \sin t, \alpha a^2 (\cos^2 t - \sin^2 t + a)) \cdot \\
 & \quad \cdot (-\sin t, \cos t, 0) a \, dt \\
 &= a^2 \int_0^{2\pi} \{-\alpha \cos^2 t \sin t - \cos t \sin t - \sin^2 t + \alpha \sin^2 t \cos t - \cos^2 t - \sin t \cos t + 0\} dt \\
 &= a^3 \int_0^{2\pi} (-\sin^2 t - \cos^2 t) dt = -2\pi a^2.
 \end{aligned}$$

ALTERNATIVELY,

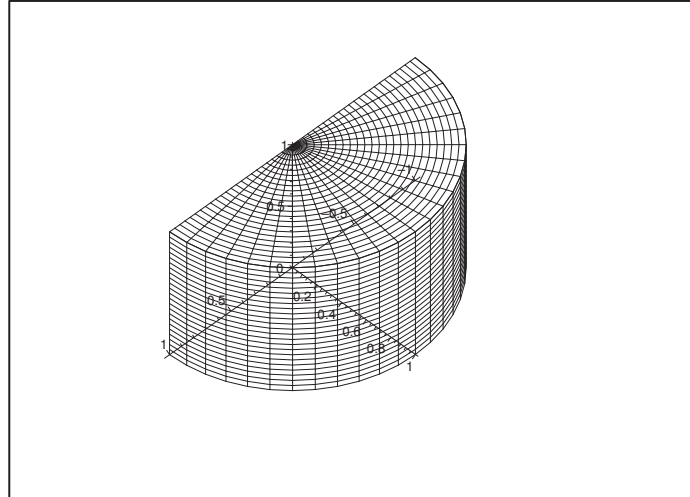
$$\mathbf{rot} \, \mathbf{V} \cdot \mathbf{e}_z = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ xz + yz & -xz - yz \end{vmatrix} = -z - z = -2z.$$

Choose \mathcal{F} as the disc $x^2 + y^2 \leq a^2$, $z = a$. Then we get by Stokes's theorem that

$$\oint_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} \, dS = \int_{\mathcal{F}} \mathbf{rot} \, \mathbf{V} \cdot \mathbf{e}_z \, dS = \int_{\mathcal{F}} (-2a) \, dS = -2a \, \text{area}(\mathcal{F}) = -2\pi a^3.$$

4) When we apply Gauß's theorem and 1), it follows that the flux is given by

$$\begin{aligned}
 \int_{\partial\Omega} \mathbf{V} \cdot \mathbf{n} \, dS &= \int_{\Omega} \text{div} \, \mathbf{V} \, d\Omega = 2\alpha \int_{\Omega} (x + y + z) \, d\Omega = 2\alpha \int_{\Omega} (y + z) \, d\Omega \\
 &= 2\alpha a \int_{-a}^a \left\{ \int_0^{\sqrt{a^2 - x^2}} y \, dy \right\} dx + 2\alpha \cdot \frac{\pi a^2}{2} \int_0^a z \, dz \\
 &= \alpha a \cdot 2 \int_0^a (a^2 - x^2) \, dx + \alpha \pi a^2 \cdot \frac{a^2}{2} = 2\alpha a \left(a^3 - \frac{1}{3} a^3 \right) + \frac{1}{2} \alpha \pi a^4 \\
 &= \frac{4}{3} \alpha a^4 + \frac{1}{2} \alpha \pi a^4 = \alpha a^4 \left(\frac{4}{3} + \frac{\pi}{2} \right).
 \end{aligned}$$

Figure 44: The body Ω for $a = 1$.

Example 5.5 Consider the space curve \mathcal{K} given by the parametric description

$$(x, y, z) = \left(1 + \cos t, \sin t, 2 \sin \frac{t}{2} \right), \quad t \in [0, 2\pi].$$

1. Find a parametric description of the tangent of \mathcal{K} at the point corresponding to $t = \frac{\pi}{2}$.
2. Show that \mathcal{K} lies on a sphere of centrum at $(0, 0, 0)$, and find an equation of the sphere.

Furthermore, consider the surface \mathcal{F} given by the parametric description

$$(x, y, z) = \left(1 + \cos t, \sin t, 2u \sin \frac{t}{2} \right), \quad (t, u) \in [0, 2\pi] \times [0, 1],$$

and the vector field $\mathbf{V}(x, y, z) = (x, y, z)$, $(x, y, z) \in \mathbb{R}^3$.

3. Find the area of \mathcal{F} .
4. Find the circulation of the vector field along the curve \mathcal{K} .

A Space curve, surface area, circulation of a vector field.

D 1) First calculate $\mathbf{r}'(t)$.

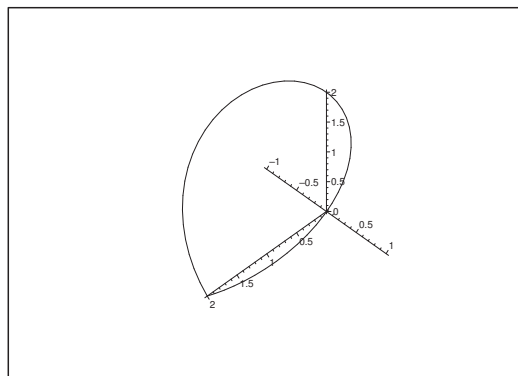
2) Show that $x^2 + y^2 + z^2 = r^2 > 0$ and find $r > 0$.

3) Compute the surface area.

4) Apply Stokes's theorem. Alternatively the circulation is computed directly as a line integral.

I 1) We find

$$\mathbf{r}'(t) = \left(-\sin t, \cos t, \cos \frac{t}{2} \right), \quad \mathbf{r}'\left(\frac{\pi}{2}\right) = \left(-1, 0, \frac{1}{\sqrt{2}} \right).$$

Figure 45: The curve \mathcal{K} .

Now, $\mathbf{r}\left(\frac{\pi}{2}\right) = (1, 1, \sqrt{2})$, thus a parametric description of the tangent is given by

$$(1, 1, \sqrt{2}) + u \left(-1, 0, \frac{1}{\sqrt{2}}\right), \quad u \in \mathbb{R}.$$

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2) Since

$$\begin{aligned} x(t)^2 + y(t)^2 + z(t)^2 &= (1 + \cos t)^2 + \sin^2 t + 4 \sin^2 \frac{t}{2} \\ &= 1 + 2 \cos t + \cos^2 t + \sin^2 t + 2(1 - \cos t) \\ &= 1 + 1 + 2 = 4 = 2^2, \end{aligned}$$

it follows that \mathcal{K} lies on the sphere of centrum $(0, 0, 0)$ and radius 2.

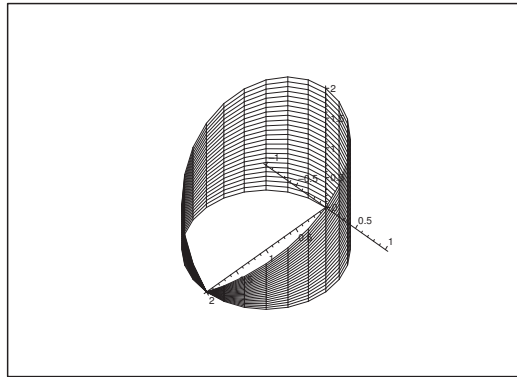


Figure 46: The surface \mathcal{F} .

3) It follows from

$$\frac{\partial \mathbf{r}}{\partial t} = \left(-\sin t, \cos t, u \cos \frac{t}{2} \right) \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial u} = \left(0, 0, 2 \sin \frac{t}{2} \right)$$

that

$$\mathbf{N}(t, u) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ -\sin t & \cos t & u \cos \frac{t}{2} \\ 0 & 0 & 2 \sin \frac{t}{2} \end{vmatrix} = 2 \sin \frac{t}{2} (\cos t, \sin t, 0),$$

thus

$$\|\mathbf{N}(t, u)\| = \left| 2 \sin \frac{t}{2} \right| \cdot 1 = 2 \sin \frac{t}{2}, \quad t \in [0, 2\pi], \quad u \in [0, 1].$$

Hence

$$\text{area}(\mathcal{F}) = \int_0^1 \left\{ \int_0^{2\pi} 2 \sin \frac{t}{2} dt \right\} = \left[-4 \cos \frac{t}{2} \right]_0^{2\pi} = 4(1 + 1) = 8.$$

4) Since $\mathbf{rot} \mathbf{V} = \mathbf{0}$, it follows by Stokes's theorem no matter how we choose the surface \mathcal{F}_1 with boundary curve \mathcal{K} that

$$\oint_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} dt = \int_{\mathcal{F}_1} \mathbf{n} \cdot \mathbf{rot} \mathbf{V} dS = 0.$$

ALTERNATIVELY we get by the definition that

$$\begin{aligned}\oint_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} \, dt &= \int_0^{2\pi} \mathbf{V} \cdot \mathbf{r}'(t) \, dt = \int_0^{2\pi} \mathbf{r}(t) \cdot \mathbf{r}'(t) \, dt \\ &= \int_0^{2\pi} \frac{d}{dt} \left(\frac{1}{2} \|\mathbf{r}(t)\|^2 \right) \, dt = \frac{1}{2} (\|\mathbf{r}(2\pi)\|^2 - \|\mathbf{r}(0)\|^2) = 0,\end{aligned}$$

because the curve is closed.

ALTERNATIVELY it is possible though extremely tedious to insert the parametric description and then reduce.

Example 5.6 Given the vector field

$$\mathbf{V}(x, y, z) = (y^3 - xz^2, -x^3 + yz^2, z^3), \quad (x, y, z) \in \mathbb{R}^3.$$

1. Find the divergence $\nabla \cdot \mathbf{V}$ and the rotation $\nabla \times \mathbf{V}$.

Let a be a positive constant, and let L denote the half spherical shell given by

$$z \geq 0, \quad a^2 \leq x^2 + y^2 + z^2 \leq 3a^2.$$

2. Find the flux of \mathbf{V} through ∂L .

Let \mathcal{C} be the circle in the plane $z = a$ of centrum $(0, 0, a)$ and radius a .

3. Find the absolute value of the circulation $\oint_{\mathcal{C}} \mathbf{V} \cdot \mathbf{t} \, ds$.

4. Check if there exists a vector field $\mathbf{W} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, such that

$$\mathbf{V} = \nabla \times \mathbf{W},$$

in the whole space.

5. Check if there exists a scalar field $F : \mathbb{R}^3 \rightarrow \mathbb{R}$, such that $\mathbf{V} = \nabla F$ in the whole space.

A Divergence, rotation, flux, circulation, vector potential, gradient field.

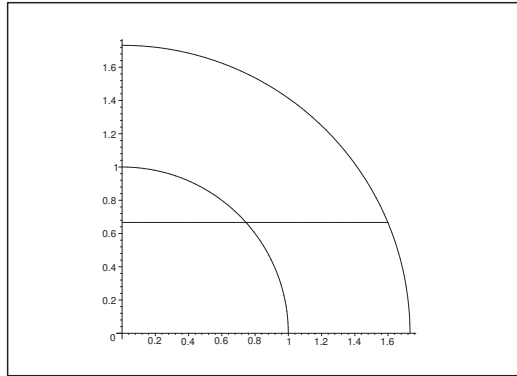
D Apply Gauß's theorem and Stokes's theorem, whenever it is possible.

I 1) We get by some very simple calculations that

$$\operatorname{div} \mathbf{V} = \nabla \cdot \mathbf{V} = -z^2 + z^2 + 3z^2 = 3z^2,$$

and

$$\nabla \times \mathbf{V} = \mathbf{rot} \, \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3 - xz^2 & -x^3 + yz^2 & z^3 \end{vmatrix} = (-2yz, -2xz, -3zx^2 - 3y^2).$$

Figure 47: The meridian cut for $a = 1$ with the cut at height z .

2) It follows from Gauß's theorem and the result of 1) that

$$\int_{\partial L} \mathbf{V} \cdot \mathbf{n} dS = \int_L \operatorname{div} \mathbf{V} d\Omega = \int_L 3z^2 d\Omega.$$

At height $z \in [0, a]$ the body L is cut in an annulus of the area

$$\pi(3a^2 - z^2) - \pi(a^2 - z^2) = 2\pi a^2.$$

At height $z \in [a, \sqrt{3}a]$ the body L is cut in a circle of area

$$\pi(3a^2 - z^2).$$

Hence by insertion,

$$\begin{aligned} \int_{\partial L} \mathbf{V} \cdot \mathbf{n} dS &= \int_L 3z^2 d\Omega = \int_0^a 3z^2 \cdot 2\pi a^2 dz + \int_a^{\sqrt{3}a} 3z^2 \pi(3a^2 - z^2) dz \\ &= 6\pi a^2 \int_0^a z^2 dz + 9\pi a^2 \int_a^{\sqrt{3}a} z^2 dz - 3\pi \int_a^{\sqrt{3}a} z^4 dz \\ &= 9\pi a^2 \int_0^{\sqrt{3}a} z^2 dz - 3\pi a^2 \int_0^a z^2 dz - 3\pi \int_a^{\sqrt{3}a} z^4 dz \\ &= 3\pi a^2 \cdot 3\sqrt{3} a^3 - \pi a^2 \cdot a^3 - \frac{3}{5} \pi (9\sqrt{3} - 1)a \\ &= \pi a^5 \left(9\sqrt{3} - 1 - \frac{27}{5} \sqrt{3} + \frac{3}{5} \right) = \frac{\pi}{5} (18\sqrt{3} - 2)a^5. \end{aligned}$$

3) Put $B = \{(x, y) \mid x^2 + y^2 \leq a^2\}$. Then we get by Stokes's theorem and the result of 1) that

$$\begin{aligned} \left| \oint_C \mathbf{V} \cdot \mathbf{t} ds \right| &= \left| \int_B \operatorname{rot} \mathbf{V} \cdot \mathbf{n} dx dy \right| = \left| \int_B (-2ya, -2xa, -3x^2 - 3y^2) \cdot (0, 0, 1) dx dy \right| \\ &= 3 \int_B (x^2 + y^2) dx dy = 3 \cdot 2\pi \int_0^a \varrho^2 \cdot \varrho d\varrho = 6\pi \cdot \frac{a^4}{4} = \frac{3\pi a^4}{2}. \end{aligned}$$

4) Since $\operatorname{div} \mathbf{V} = 3z^2 \neq 0$ for $z \neq 0$, there exists no vector potential \mathbf{W} of \mathbf{V} in all of the space.

5) It follows from

$$\frac{\partial V_1}{\partial y} = 3y^2 \quad \text{and} \quad \frac{\partial V_2}{\partial x} = -3x^2,$$

that

$$\frac{\partial V_1}{\partial y} \neq \frac{\partial V_2}{\partial x} \quad \text{for } (x, y) \neq (0, 0),$$

thus \mathbf{V} is not a gradient field, and there exists no integral F of \mathbf{V} .

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Example 5.7 Given the vector fields

$$\mathbf{U}(x, y, z) = (z^2 + y \cos x, x^2 + \sin x, y^2), \quad \mathbf{V}(x, y, z) = (y, z, x),$$

in the space \mathbb{R}^3 .

1) Find the divergence and the rotation of both vector fields.

2) Find the flux of \mathbf{U} through the surface of the cube

$$\{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}.$$

3) Let a be a positive constant. Find the circulation of \mathbf{V} along the circle in the (X, Z) -plane of center at $(a, 0, 2a)$ and radius a . Choose an orientation of the circle.

4) Find a vector potential for \mathbf{V} .

A Vector analysis.

D Apply Gauß's theorem and Stokes's theorem.

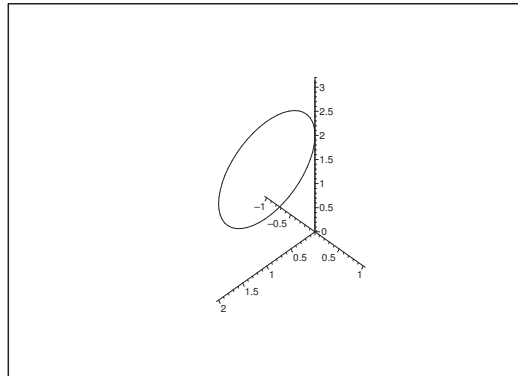


Figure 48: The circle of 3).

I 1) By simple calculations,

$$\operatorname{div} \mathbf{U} = -y \sin x, \quad \operatorname{div} \mathbf{V} = 0,$$

$$\operatorname{rot} \mathbf{U} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 + y \cos x & x^2 + \sin x & y^2 \end{vmatrix} = (2y, 2z, 2x) = 2\mathbf{V},$$

$$\operatorname{rot} \mathbf{V} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = (-1, -1, -1).$$

In particular, $\mathbf{V} = \nabla \times \left(\frac{1}{2}\mathbf{U}\right)$, thus $\frac{1}{2}\mathbf{U}$ is a vector potential of \mathbf{V} (cf. 4)).

2) According to Gauß's theorem the flux of \mathbf{U} through ∂T is given by

$$\begin{aligned}\int_{\partial T} \mathbf{U} \cdot \mathbf{n} dS &= \int_T \operatorname{div} \mathbf{U} d\Omega = \int_0^1 \left\{ \int_0^1 \left\{ \int_0^1 (-y \sin x) dx \right\} dy \right\} dz \\ &= 1 \cdot \left[\frac{y^2}{2} \right]_0^1 \cdot [\cos x]_0^1 = -\frac{1}{2}(1 - \cos 1).\end{aligned}$$

3) According to Stokes's theorem,

$$\oint_{\partial A} \mathbf{t} \cdot \mathbf{V} ds = \int_A \mathbf{n} \cdot \operatorname{rot} \mathbf{V} dS = \int_A (0, -1, 0) \cdot (-1, -1, -1) dS = \operatorname{area}(A) = \pi a^2.$$

4) According to the result of 1), the field $\frac{1}{2}\mathbf{U}$ is a vector potential of \mathbf{V} .

Example 4.8 Let $p \in \mathbb{R}$ and $b \in \mathbb{R}_+$ be constants. consider the circle \mathcal{K} given by $x^2 + y^2 = b^2$, $z = p$; the circle is run through in that direction which form a right hand turn with the Z -axis. Furthermore, consider the vector field

$$\mathbf{W}(x, y, z) = \left(\frac{yz}{\sqrt{x^2 + y^2}}, \frac{-xz}{\sqrt{x^2 + y^2}}, \sqrt{x^2 + y^2} \right), \quad (x, y) \neq (0, 0).$$

Denote the circulation of \mathbf{W} along the oriented circle \mathcal{K} by $C(b, p)$.

1) Show that $C(b, p) = -2\pi pb$.

2) Let $\mathbf{V} = \operatorname{rot} \mathbf{W}$. Show that

$$\mathbf{V}(x, y, z) = \frac{1}{\sqrt{x^2 + y^2}}(x + y, y - x, -z), \quad (x, y) \neq (0, 0).$$

3) Show that \mathbf{W} is not a gradient field.

4) Show that \mathbf{W} has zero divergence.

5) Let \mathcal{O} be the surface of revolution which is introduced in Calculus 2c-8, **Example 1.13**. Find the flux

$$\int_{\mathcal{O}} \mathbf{V} \cdot \mathbf{n} dS,$$

where we must choose an orientation of \mathcal{O} .

A Circulation, rotation, gradient field, divergence, flux.

D The circulation can be computed in various ways. The computation of the flux has also some variants.

I 1) We have two variants.

First variant. *The definition of the circulation as a line integral.*

We use the following parametric description of the circle \mathcal{K} ,

$$\mathbf{r}(t) = (x, y, z) = (b \cdot \cos t, b \cdot \sin t, p), \quad t \in [0, 2\pi].$$

Then

$$\mathbf{r}'(t) = b(-\sin t, \cos t, 0),$$

and the circulation is according to the definition given by

$$\begin{aligned} C(b, p) &= \int_{\mathcal{K}} \mathbf{W} \cdot \mathbf{t} \, ds = \int_0^{2\pi} \left(\frac{b \sin t \cdot p}{b}, -\frac{b \cos t \cdot p}{b}, b \right) \cdot b(-\sin t, \cos t, 0) \, dt \\ &= -pb \int_0^{2\pi} \{\sin^2 t + \cos^2 t + 0\} \, dt = -2\pi pb. \end{aligned}$$

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Second variant. *Stokes's theorem.*

An application of Stokes's theorem gives

$$\int_{\mathcal{K}} \mathbf{W} \cdot \mathbf{t} \, ds = \int_{\mathcal{F}} \mathbf{rot} \, \mathbf{W} \cdot \mathbf{n} \, dS,$$

where \mathcal{F} is the disc at height $z = p$ and radius b , and where the unit normal vector is parallel to the Z -axis.

The unit normal is trivially $\mathbf{n} = (0, 0, 1)$. then by 2),

$$\mathbf{rot} \, \mathbf{W} = \mathbf{V}.$$

By applying the expression of \mathbf{V} , we obtain in polar coordinates

$$\int_{\mathcal{K}} \mathbf{W} \cdot \mathbf{t} \, ds = \int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n} \, dS = \int_{\mathcal{F}} \left(-\frac{p}{\sqrt{x^2 + y^2}} \right) dS = -p \int_0^{2\pi} \left\{ \int_0^b \frac{1}{\varrho} \cdot \varrho \, d\varrho \right\} d\varphi = -2\pi pb.$$

2) Let $(x, y) \neq (0, 0)$. Then

$$\begin{aligned} \mathbf{V} = \mathbf{rot} \, \mathbf{W} = \nabla \times \mathbf{W} &= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{yz}{\sqrt{x^2 + y^2}} & \frac{-xz}{\sqrt{x^2 + y^2}} & \sqrt{x^2 + y^2} \end{vmatrix} \\ &= \begin{pmatrix} \frac{y}{\sqrt{x^2 + y^2}} - \left(-\frac{x}{\sqrt{x^2 + y^2}} \right) \\ \frac{y}{\sqrt{x^2 + y^2}} - \frac{x}{\sqrt{x^2 + y^2}} \\ -\frac{z}{\sqrt{x^2 + y^2}} + \frac{x^2 z}{(\sqrt{x^2 + y^2})^3} - \frac{z}{\sqrt{x^2 + y^2}} + \frac{y^2 z}{(\sqrt{x^2 + y^2})^3} \end{pmatrix} \\ &= \frac{1}{\sqrt{x^2 + y^2}} \begin{pmatrix} y + x \\ y - x \\ -2z + \frac{x^2 z + y^2 z}{x^2 + y^2} \end{pmatrix} = \frac{1}{\sqrt{x^2 + y^2}} (x + y, y - x, -z). \end{aligned}$$

3) Suppose \mathbf{W} was a gradient field, ∇F . Then

$$\mathbf{V} = \nabla \times \mathbf{W} = \nabla \times \nabla F = \mathbf{0}.$$

But $\mathbf{V} \neq \mathbf{0}$, thus we conclude that \mathbf{W} is not a gradient field.

4) By just computing,

$$\begin{aligned} \operatorname{div} \mathbf{W} &= \frac{\partial}{\partial x} \left(\frac{yz}{\sqrt{x^2 + y^2}} \right) - \frac{\partial}{\partial y} \left(\frac{xz}{\sqrt{x^2 + y^2}} \right) + \frac{\partial}{\partial z} \left(\sqrt{x^2 + y^2} \right) \\ &= -\frac{xyz}{(x^2 + y^2)^{3/2}} - \left(-\frac{xyz}{(x^2 + y^2)^{3/2}} \right) + 0 = 0. \end{aligned}$$

ALTERNATIVELY it follows that if \mathbf{U} is defined by

$$3\mathbf{U} = \frac{z^2}{\sqrt{x^2 + y^2}} (-x, -y, 0) + \sqrt{x^2 + y^2} (-y, x, z), \quad (x, y) \neq (0, 0),$$

then

$$\mathbf{W} = \text{rot } \mathbf{U} = \nabla \times \mathbf{U},$$

and thus

$$\text{div } \mathbf{W} = \nabla \cdot (\nabla \times \mathbf{U}) = 0.$$

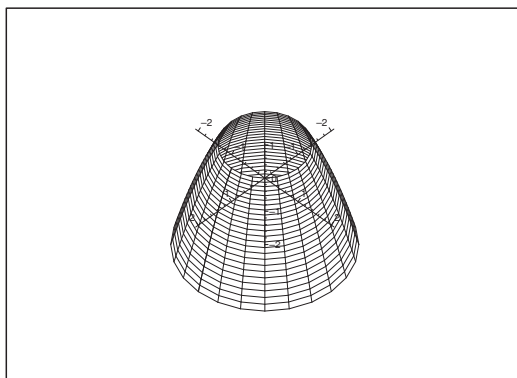


Figure 49: The surface \mathcal{O} . The upper boundary curve \mathcal{K}_1 is oriented as a left hand screw, while the lower boundary curve \mathcal{K}_2 is oriented as a right hand screw. Hence the normal vector field on \mathcal{O} is everywhere pointing away from the Z -axis.

- 5) Choose the orientation on \mathcal{O} as described in the caption of the figure. Then $\delta\mathcal{O} = \mathcal{K}_2 - \mathcal{K}_1$, where the minus sign in front of \mathcal{K}_1 means that this circle is run through in the opposite direction of the usual one, i.e. as a left handed screw.

There are two variants.

1. variant. *Stokes's theorem combined with 1).*

We get by Stokes's theorem,

$$\begin{aligned} \int_{\mathcal{O}} \mathbf{V} \cdot \mathbf{n} dS &= \int_{\mathcal{O}} (\nabla \times \mathbf{W}) \cdot \mathbf{n} dS = \oint_{\delta\mathcal{O}} \mathbf{W} \cdot \mathbf{t} ds = - \int_{\mathcal{K}_1} \mathbf{W} \cdot \mathbf{t} ds + \int_{\mathcal{K}_2} \mathbf{W} \cdot \mathbf{t} ds \\ &= -C(a, a) + C(2a, -2a) = +2\pi a \cdot a + (+2\pi \cdot 2a \cdot 2a) = 10\pi a^2. \end{aligned}$$

Second variant. *Surface integral.*

The meridian curve has the equation

$$z = 2a - \frac{\varrho^2}{a},$$

so we conclude that the tangent vector is $\left(1, -\frac{2\rho}{a}\right)$. Hence the normal vector $\mathbf{N} = (2\rho, a)$, and thus the unit normal vector

$$\mathbf{n} = \frac{1}{\sqrt{a^2 + 4\rho^2}} (2\rho, a).$$

Then the outgoing unit normal vector field of the surface \mathcal{O} is

$$\mathbf{n}(\rho, \varphi) = \frac{1}{\sqrt{a^2 + 4\rho^2}} (2\rho \cos \varphi, 2\rho \sin \varphi, a).$$

We have on \mathcal{O} ,

$$\begin{aligned} \mathbf{V} &= \frac{1}{\sqrt{x^2 + y^2}} (x + y, y - x, -z) \\ &= \frac{1}{\rho} \left(\rho \cos \varphi + \rho \sin \varphi, \rho \sin \varphi - \rho \cos \varphi, -2a + \frac{\rho^2}{a} \right) \\ &= \left(\cos \varphi + \sin \varphi, \sin \varphi - \cos \varphi, \frac{\rho}{a} - 2 \frac{a}{\rho} \right), \end{aligned}$$

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thence the integrand over \mathcal{O} is written

$$\begin{aligned} f(x, y, z) &= \mathbf{V} \cdot \mathbf{n} = \frac{1}{\sqrt{a^2 + 4\rho^2}} \left\{ 2\rho (\cos^2 \varphi + \cos \varphi \sin \varphi + \sin^2 \varphi - \cos \varphi \sin \varphi) + \rho - 2 \frac{a^2}{\rho} \right\} \\ &= \frac{1}{\sqrt{a^2 + 4\rho^2}} \left\{ 3\rho - 2 \frac{a^2}{\rho} \right\}. \end{aligned}$$

Finally, by inserting into a known formula of the surface integral over surfaces of revolution, we get

$$\begin{aligned} \int_{\mathcal{O}} \mathbf{V} \cdot \mathbf{n} dS &= \int_a^{2a} \left\{ \int_0^{2\pi} \frac{1}{\sqrt{a^2 + 4\rho^2}} \left\{ 3\rho - 2 \frac{a^2}{\rho} \right\} d\varphi \right\} \rho \sqrt{1 + \frac{4\rho^2}{a^2}} d\rho \\ &= \frac{2\pi}{a} \int_a^{2a} (3\rho^2 - 2a^2) d\rho = \frac{2\pi}{a} [\rho^3 - 2a^2\rho]_a^{2a} \\ &= \frac{2\pi}{a} \{8a^3 - 4a^3 - a^3 + 2a^3\} = 10\pi a^2. \end{aligned}$$

Example 5.9 Let \mathcal{F} be one eighth of a sphere given by

$$x^2 + y^2 + z^2 = a^2, \quad x \geq 0, \quad y \geq 0, \quad z \geq 0,$$

where a is a positive constant. Thus the boundary curve $\delta\mathcal{F}$ is composed of three circular arcs.

Also, consider the vector field

$$\mathbf{V}(x, y, z) = (ay + yz, -ax + zx, z^2 - 2xy), \quad (x, y, z) \in \mathbb{R}^3.$$

- 1) Find the rotation $\nabla \times \mathbf{V}$.
- 2) Show that \mathbf{V} is not a gradient field.
- 3) Find the circulation

$$\oint_{\delta\mathcal{F}} \mathbf{V} \cdot \mathbf{t} ds,$$

where we choose an orientation of $\delta\mathcal{F}$.

A Rotation, circulation, Stokes's theorem.

D Sketch a figure. Apply Stokes's theorem.

I 1) The rotation is

$$\begin{aligned} \mathbf{rot} \mathbf{V} &= \nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ay + yz & -ax + zx & z^2 - 2xy \end{vmatrix} \\ &= (-2x - x, y + 2y, -a + zaz) = (3x, 3y, -2a). \end{aligned}$$

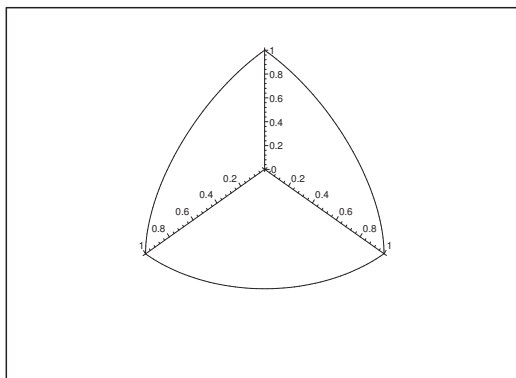


Figure 50: The surface \mathcal{F} and the boundary curve $\delta\mathcal{F}$ for $a = 1$. On the surface \mathcal{F} the unit normal vector field is always directed away from $(0, 0, 0)$, and the curve $\delta\mathcal{F}$ is oriented correspondingly, i.e. from the X -axis towards the y -axis, then towards the z -axis and finally back to the x -axis.

- 2) It follows from $\mathbf{rot}\ \mathbf{V} \neq \mathbf{0}$ that \mathbf{V} is not a gradient field.
- 3) Choose the orientation of $\delta\mathcal{F}$ as described on the figure. Then the unit normal vector field on \mathcal{F} is pointing outwards, i.e.

$$\mathbf{n} = \frac{1}{a}(x, y, z), \quad \text{for } (x, y, z) \in \mathcal{F}.$$

Applying Stokes's theorem we conclude that the circulation along $\delta\mathcal{F}$ is

$$\begin{aligned} \oint_{\delta\mathcal{F}} \mathbf{V} \cdot \mathbf{t}\ ds &= \int_{\mathcal{F}} \mathbf{n} \cdot \mathbf{rot}\ \mathbf{V}\ dS = \int_{\mathcal{F}} \frac{1}{a}(x, y, z) \cdot (-3x, 3y, -2a)\ dS \\ &= \frac{1}{a} \int_{\mathcal{F}} (-3x^2 + 3y^2 - 2az)\ dS = -2 \int_{\mathcal{F}} z\ dS, \end{aligned}$$

where it follows by the symmetry that

$$\int_{\mathcal{F}} x^2\ dS = \int_{\mathcal{F}} y^2\ dS.$$

The following computations can be given in various variants.

First variant. If we first (i.e. innermost) at height z and denote the circle by ℓ , then

$$\begin{aligned} \oint_{\delta\mathcal{F}} \mathbf{V} \cdot \mathbf{t}\ ds &= -2 \int_{\mathcal{F}} z\ dS = -2 \int_{\ell} z \cdot \frac{\pi}{2} \cdot \sqrt{a^2 - z^2}\ ds \\ &= -\pi \int_{\ell} z \sqrt{a^2 - z^2}\ ds = \left(-\pi \int_{\ell} z \varrho\ ds \right). \end{aligned}$$

Using the parametric description

$$\varrho = a \cos \varphi, \quad z = a \sin \varphi,$$

of ℓ we get $ds = a\ d\varphi$, and the computations continue as follows,

$$\begin{aligned} \oint_{\delta\mathcal{F}} \mathbf{V} \cdot \mathbf{t}\ ds &= -\pi \int_0^{\frac{\pi}{2}} a \sin \varphi \cdot a \cos \varphi \cdot a\ d\varphi = -\pi a^3 \int_0^{\frac{\pi}{2}} \sin \varphi \cos \varphi\ d\varphi \\ &= -\pi a^3 \left[\frac{\sin^2 \varphi}{2} \right]_0^{\frac{\pi}{2}} = -\frac{\pi}{2} a^3. \end{aligned}$$

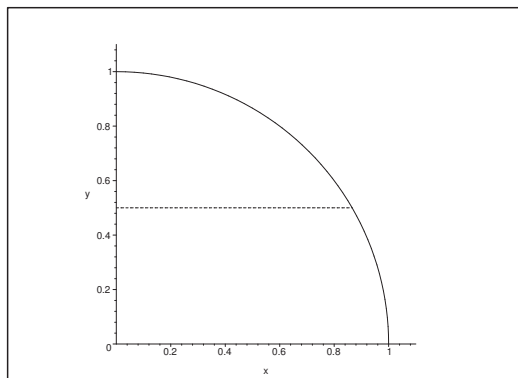


Figure 51: The meridian cut of \mathcal{F} for $a = 1$. We have at height z that $\rho = \sqrt{a^2 - z^2}$.

Second variant. The surface \mathcal{F} is described in spherical coordinates by

$$\begin{cases} x = a \sin \theta \cos \varphi, \\ y = a \sin \theta \sin \varphi, \\ z = a \cos \theta, \end{cases} \quad \begin{cases} \theta \in [0, \frac{\pi}{2}], \\ \varphi \in [0, \frac{\pi}{2}], \end{cases} \quad \text{weight: } a^2 \sin \theta,$$

hence by insertion,

$$\begin{aligned} \oint_{\delta\mathcal{F}} \mathbf{V} \cdot \mathbf{t} \, ds &= - \int_{\mathcal{F}} 2z \, dS = -2 \int_0^{\frac{\pi}{2}} \left\{ \int_0^{\frac{\pi}{2}} a \cos \theta \cdot a^2 \sin \theta \, d\theta \right\} d\varphi \\ &= -2 \cdot \frac{\pi}{2} \cdot a^3 \left[\frac{\sin^2 \varphi}{2} \right]_0^{\frac{\pi}{2}} = -\frac{\pi}{2} a^3. \end{aligned}$$

Third variant. *Direct computation of the line integrals without the use of Stokes's theorem.*

First note that the boundary curve $\delta\mathcal{F}$ is composed of the subcurves:

Γ_1 : $(x, y, z) = (a \cos \varphi, a \sin \varphi, 0)$, $\varphi \in [0, \frac{\pi}{2}]$, with the unit tangent vector $\mathbf{t} = (-\sin \varphi, \cos \varphi, 0)$, and the line element $ds = a \, d\varphi$.

Γ_2 : $(x, y, z) = (0, a \cos \varphi, a \sin \varphi)$, $\varphi \in [0, \frac{\pi}{2}]$, with the unit tangent vector $\mathbf{t} = (0, -\sin \varphi, \cos \varphi)$, and the line element $ds = a \, d\varphi$,

Γ_3 : $(x, y, z) = (a \sin \varphi, 0, a \cos \varphi)$, $\varphi \in [0, \frac{\pi}{2}]$, with the unit tangent vector $\mathbf{t} = (\cos \varphi, 0, -\sin \varphi)$ and the line element $ds = a \, d\varphi$.

We get by insertion,

$$\oint_{\delta\mathcal{F}} \mathbf{V} \cdot \mathbf{t} \, ds = \int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\Gamma_3} (1t + yz, -ax + zx, z^2 - 2xy) \cdot \mathbf{t} \, ds.$$

The integrals are computed one at a time,

$$\begin{aligned}\int_{\Gamma_1} \mathbf{V} \cdot \mathbf{t} \, ds &= \int_0^{\frac{\pi}{2}} (a^2 \sin \varphi, -a^2 \cos \varphi, -2a^2 \sin \varphi \cos \varphi) \cdot (-\sin \varphi, \cos \varphi, 0) a \, d\varphi \\ &= \int_0^{\frac{\pi}{2}} a^3 (-\sin^2 \varphi - \cos^2 \varphi) \, d\varphi = -\frac{\pi}{2} a^3,\end{aligned}$$

$$\begin{aligned}\int_{\Gamma_2} \mathbf{V} \cdot \mathbf{t} \, ds &= \int_0^{\frac{\pi}{2}} (a^2 \cos \varphi + a^2 \sin \varphi \cos \varphi, 0, a^2 \sin^2 \varphi) \cdot (0, -\sin \varphi, \cos \varphi) a \, d\varphi \\ &= \int_0^{\frac{\pi}{2}} a^3 \sin^2 \varphi \cos \varphi \, d\varphi = 3 \left[\frac{\sin^3 \varphi}{3} \right]_0^{\frac{\pi}{2}} = \frac{1}{3} a^3,\end{aligned}$$

$$\begin{aligned}\int_{\Gamma_3} \mathbf{V} \cdot \mathbf{t} \, ds &= \int_0^{\frac{\pi}{2}} (0, -a^2 \sin \varphi + a^2 \sin \varphi \cos \varphi, a^2 \cos^2 \varphi) \cdot (\cos \varphi, 0, -\sin \varphi) a \, d\varphi \\ &= \int_0^{\frac{\pi}{2}} (-a^3 \cos^2 \varphi \sin \varphi) \, d\varphi = a^3 \left[\frac{\cos^3 \varphi}{3} \right]_0^{\frac{\pi}{2}} = -\frac{1}{3} a^3.\end{aligned}$$

Summarizing,

$$\oint_{\delta \mathcal{F}} \mathbf{V} \cdot \mathbf{t} \, ds = -\frac{\pi}{2} a^3 + \frac{1}{3} a^3 - \frac{1}{3} a^3 = -\frac{\pi}{2} a^3.$$

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Example 5.10 Consider the vector field

$$\mathbf{V}(x, y, z) = (xz, yz + xz, 2xz - yz), \quad (x, y, z) \in \mathbb{R}^3.$$

1. Find the divergence $\nabla \cdot \mathbf{V}$ and the rotation $\nabla \times \mathbf{V}$.

Let A denote the half ball given by

$$x^2 + y^2 + z^2 \leq c^2, \quad z \geq 0,$$

where c is a positive constant, and let \mathbf{n} be the outwards unit normal vector of the surface ∂A .

2. Find the flux

$$\Phi = \int_{\partial A} \mathbf{V} \cdot \mathbf{n} dS.$$

3. The surface ∂A is the union of a disc \mathcal{F}_1 and a half sphere \mathcal{F}_2 . Find the fluxes

$$\Phi_1 = \int_{\mathcal{F}_1} \mathbf{V} \cdot \mathbf{n} dS \quad \text{and} \quad \Phi_2 = \int_{\mathcal{F}_2} \mathbf{V} \cdot \mathbf{n} dS.$$

Let \mathcal{K} denote a circle in the plane of equation $z = b$. We denote the centrum of the circle by (x_0, y_0, b) , and its radius is called a .

4. Choose an orientation of the circle \mathcal{K} . Then find the circulation

$$C = \oint_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} ds.$$

A Divergence, rotation, flux, circulation.

D Follow the guidelines which give the simplest variant.

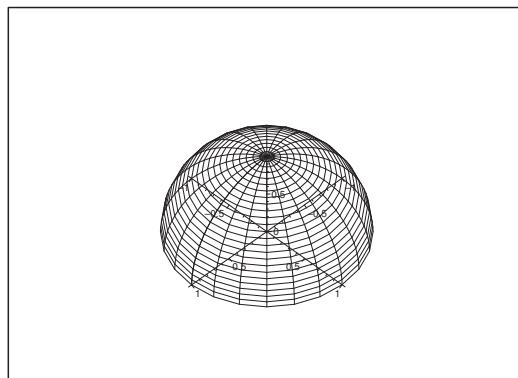


Figure 52: The half ball A for $c = 1$.

I 1) By just computing we get

$$\operatorname{div} \mathbf{V} = \nabla \cdot \mathbf{V} = z + z + 2x - y = 2x - y + 2z$$

and

$$\begin{aligned} \operatorname{rot} \mathbf{V} &= \nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & yz + xz & 2xz - yz \end{vmatrix} \\ &= (-z - y - x, x - 2x, z) = (-x - y - z, -x, z). \end{aligned}$$

2) Then by Gauß's theorem,

$$\begin{aligned} \Phi &= \int_{\partial A} \mathbf{V} \cdot \mathbf{n} dS = \int_A \operatorname{div} \mathbf{V} d\Omega = \int_A (2x - y + 2z) d\Omega \\ &= 0 + 0 + 2 \int_A z d\Omega = 2 \cdot \frac{\pi}{4} c^4 = \frac{\pi}{2} c^4. \end{aligned}$$

3) Now $\mathbf{n} = (0, 0, -1)$ on \mathcal{F}_1 , where also $z = 0$. Hence

$$\Phi_1 = \int_{\mathcal{F}_1} (0, 0, 0) \cdot \mathbf{n} dS = 0.$$

Then apply the result of 2) and that $\Phi = \Phi_1 + \Phi_2$, to get

$$\Phi_2 = \Phi - \Phi_1 = \frac{\pi}{2} c^4.$$

4) Choose the orientation such that the projection onto the (X, Y) -plane has a positive orientation. Then the corresponding unit normal vector is $\mathbf{n} = (0, 0, 1)$.

By Stokes's theorem, the circulation along \mathcal{K} (which encircles the disc B) is given by

$$\begin{aligned} C &= \oint_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} ds = \int_B \mathbf{n} \cdot \nabla \times \mathbf{V} dS = \int_B (0, 0, 1) \cdot (-x - y - b, -x, -b) dS \\ &= -b \int_B dS = -b \cdot \operatorname{area}(B) = -b \cdot \pi \cdot a^2. \end{aligned}$$